Average consensus and stability analysis in networked dynamic systems

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Abstract

This paper provides protocols for finite-time average consensus and finite-time stability of systems with controlled nonlinear dynamics in network under undirected fixed topology. Each node's state is a high dimensional vector as a solution of highly nonlinear first-order dynamics with and without drift terms. Under the proposed interaction rules, agreements as a common average value or an average trajectory are reached, solving finite-time average consensus and the multi-system equilibrium is controlled leading to the finite-time stability of each system origin. Sufficient conditions are achieved using the Lyapunov techniques and the graph theory. In networked dynamic systems, the theoretical results of the paper cover a large class of underactuated autonomous systems as formation flight, multi-vehicle coordination and heterogenous multi-system behaviors. Some examples are introduced in simulation which approve the proposed protocols.

Key words: Finite-time average consensus; finite-time stability; multi-system dynamics.

1 Introduction

For cooperative tasks using multi-agent groups, the presence of a large number of autonomous dynamical systems in industry requires interrelationships between distributed control parameters which are designed at a first step to manage each agent separately. Thus, in coordination of a team of autonomous agents, the communication of sensors is fundamental in many distributed control systems. For many applications the main challenges in cooperative design for a group of agents is to meet some objectives such that the rendezvous problem of multi-vehicle, control of training, flocking, attitude synchronization and the fusion of sensors. A coherent movement in masses is called consensus. Thus, the problem of consensus plays a central role in study of multi-agent systems. In recent years this paradigm has introduced in multi-agent systems witnessed dramatic advances of various distributed strategies that achieve agreements. In [Vicsek, 1995] the authors proposed a simple but interesting discrete-time model of finite agents all moving in the plane. Each agent's motion is updated using a local rule based on its own state and the states of its neighbors. Jadbabaie *et al.* [Jadabaie, 2003] provided a theoretical explanation of the consensus property of the Vicsek model by using graph theory and nonnegative matrix theory. For this model each agent's set of neighbors changes with time as system evolves. Consequently, many seemingly different problems that involve inter-connection of dynamic systems in various areas of science and engineering happen to be closely related to consensus problems for multi-agent systems. The existing connections are presented by Olfati-Saber in [Olfati-Saber, 2007] with application to linear dynamics in network in studying of multi-system behaviors.

The theoretical framework for posing and solving consensus problems for networked dynamic systems was introduced by Olfati-Saber and Murray in [Olfati-Saber, 2003] and [Olfati-Saber, 2004] (builded on Fax's works [Fax, 2001] [Fax, 2004]). Under dynamically changing interaction topologies, Ren and

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Beard [Ren, 2005] extended the results of Jadbabaie [Jadabaie, 2003].

Various finite-time stabilizing control laws have been proposed using continuous state feedback and output feedback controllers Bhat *et al.* [Bhat, 2000]. Furthermore, the finite-time control design has been extended to n^{th} order systems with both parametric and dynamic uncertainties [Hong, 2006]. Although the finite-time design is generally more difficult than the asymptotically stabilizing control due to the lack of effective analysis tools. Also, the non-smooth finite-time control synthesis can improve the system behaviors in some aspects like high-speed, control accuracy, and disturbance- rejection. Therefore, it is not surprising that finite-time control ideas have been applied to multi-agent systems with first-order agent dynamics using gradient flow and Lyapunov function [Cortes, 2006]-[Xiao, 2006].

Finite-time consensus firstly was studied by Cortes [Cortes, 2006], where a non-smooth consensus algorithm is proposed. In the same filed [Hui, 2008], and in [Xiao, 2009] authors proposed a continuous nonlinear consensus algorithm to guarantee the finite-time stability under an undirected fixed interaction graph. Wang and Xiao in [Wang, 2010] suggest an improvement to the proposed algorithm proposed in [Hui, 2008]. The new algorithm proposed in [Wang, 2010] is able to guarantee finite-time consensus under an undirected switching interaction and a directed fixed interaction graph when each strongly connected component of the topology is detail-balanced. In [Yougcan, 2011], the authors study finite-time consensus for second order dynamics with inherent nonlinear dynamics under an undirected fixed interaction graph.

In networked dynamic systems, finite-time consensus problems that have been solved so far are mostly only for simple agents like particle behaviors as first or second order dynamics. In [Zoghlami, 2013] and [Zoghlami, 2014], the authors treated finite-time consensus for highly nonlinear dynamic systems in network affine in control inputs. Such a system is described by a nonlinear first-order ordinary differential relations.

While an interesting topic in consensus problem is the average consensus problem such that the states of all the agents converge asymptotically or in finite time to the average of their initial states under a networked interaction protocol, one cites the results in [Zhu, 2010] [Fangcui, 2011] [Shahram, 2012] [Shuai, 2013] [Liu, 2007], our work consists to extend these results and propose protocols for nonlinear dynamic systems in network expected to reach an agreement that can be a predefined average value or an average trajectory. Moreover, we will make difference between consensus and stability protocols in treating the equilibrium stability of the designed multi-system dynamics.

The paper is organized as follows. Some preliminaries results, the problem statement, and the finite-time average consensus protocol are formulated in section 2. In section 3 one solves a finite-time average consensus of multi-system without drift terms. The finite-time average consensus of multi-system with drift is detailed in section 4. Finally, illustrative examples are presented in section 5.

2 Preliminaries and problem formulation

Throughout this paper, we use $\mathbb{R}$ to denote the set of real number. $\mathbb{R}^n$ is the n-dimensional real vector space and $\|.\|$ denotes the Euclidian norm. $\mathbb{R}^{n\times n}$ is the set of $n\times n$ matrices. $diag\{m_1, m_2, ..., m_n\}$ denotes a $n\times n$ diagonal matrix. $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. The symbol $\otimes$ is the Kronecker product of matrices. We use $sgn(.)$ to denote the signum function. For a scalar x, note that $\varphi_\alpha(x) = sgn(x)|x|^\alpha$. We use $x^i = (x_1^i, ..., x_n^i)^T \in \mathbb{R}^n$, $\mathbf{x} = (x^1, ..., x^N)^T$ to denote the vector in $\mathbb{R}^{nN}$. Let $\phi_\alpha(x^i) = (\varphi_\alpha(x_1^i), ..., \varphi_\alpha(x_n^i))^T$ with $\phi_\alpha(\mathbf{x}) = (\phi_\alpha(x^i), ..., \phi_\alpha(x^N))^T$. Let $\mathbf{1}_n = (1, ..., 1)^T$. The exponent T is the transpose.

2.1 Graph theory

In this subsection, we introduce some basic concepts in algebraic graph theory for multi-agent networks. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a directed graph, where $\mathcal{V} = \{1, 2, ..., n\}$ is the set of nodes, node i represents the ith agent, $\mathcal{E}$ is the set of edges, and an edge in $\mathcal{G}$ is denoted by an ordered pair (i, j). $(i, j) \in \mathcal{E}$ if and only if the ith agent can send information to the jth agent directly.

$A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements, where $a_{ij} > 0$ if there is an edge between the ith agent and jth agent and $a_{ij} = 0$ otherwise. Moreover, if $A^T = A$, then $\mathcal{G}$ is also called an undirected graph. In this paper, we will refer to graphs whose weights take values in the set$\{0, 1\}$ as binary and those graphs whose adjacency matrices are symmetric as symmetric. Let $D = diag\{d_1, ..., d_n\} \in \mathbb{R}^{n\times n}$ be a diagonal matrix, where $d_i = \sum_{j=1}^{n} a_{ij}$ for $i = 0, 1, ..., n$. Hence, we define the Laplacian of the weighted graph

$$L = D - A \in \mathbb{R}^{n\times n}$$

The undirected graph is called connected if there is a path between any two vertices of the graph.

2.2 Some useful lemmas

Our main results are guided by the following Lemmas. The reader may find more details in the associated references.

Lemma 1 *[Bhat, 2000]. Consider the system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, $f(0) = 0$, $\boldsymbol{x} \in \mathbb{R}^n$, there exist a positive definite continuous function $V(\boldsymbol{x}) : U \subset \mathbb{R}^n \to \mathbb{R}$, real numbers $c > 0$ and $\alpha \in]0, 1[$, and an open neighborhood $U_0 \subset U$ of the* *origin such that $\dot{V} + c(V(\boldsymbol{x}))^{\alpha} \leq 0$, $\boldsymbol{x} \in U_0 \backslash \{0\}$. Then $V(\boldsymbol{x})$ converges to zero in finite time. In addition, the finite settling time T_* satisfies $T_* \leq \dfrac{V(\boldsymbol{x}(0))^{1-\alpha}}{c(1-\alpha)}$.*

Lemma 2 *[Olfati-Saber, 2004]. For a connected undirected graph $\mathcal{G}$, the Laplacian matrix L of $\mathcal{G}$ has the following properties, $\boldsymbol{x}^T L \boldsymbol{x} = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x_i - x_j)^2$, which implies that L is positive semi-definite. 0 is a simple eigenvalue of L and $\boldsymbol{1}$ is the associated eigenvector. Assume that the eigenvalues of L are denoted by $0, \lambda_2, ..., \lambda_n$ satisfying $0 \leq \lambda_2 \leq ... \leq \lambda_n$. Then the second smallest eigenvalue satisfies $\lambda_2 > 0$. Furthermore, if $\boldsymbol{1}^T \boldsymbol{x} = 0$, then $\boldsymbol{x}^T L \boldsymbol{x} \geq \lambda_2 \boldsymbol{x}^T \boldsymbol{x}$.*

Lemma 3 *[Hardy, 1952]. Let $x_1, x_2, ..., x_n \geq 0$ and $0 < p \leq 1$. Then $(\sum_{i=1}^{n} x_i)^p \leq \sum_{i=1}^{n} x_i^p \leq n^{1-p}(\sum_{i=1}^{n} x_i)^p$.*

2.3 Problem statements

We solve the finite-time average consensus and stability of two type of models in networked dynamic systems affine in control inputs. The first type is given by equation (1) which describes a controlled dynamic system without drift term. The second type is represented by relation (2) which is clearly a controlled dynamic system with drift term $f^i(x^i)$. Let consider a group of N high-dimensional agents where each agent's behavior is described by a controlled nonlinear model without drift Σ_1 represented by the controlled dynamic (1) and system Σ_2 with drift as shown by the controlled dynamic (2), $\forall i \in \mathcal{I} = \{1, ..., N\}$

$$\Sigma_1 : \quad \dot{x}^i = B(x^i)u^i \tag{1}$$

and

$$\Sigma_2 : \quad \dot{x}^i = f^i(x^i) + B(x^i)u^i \tag{2}$$

where $x^i \in \mathbb{R}^n$, $x^i = [x_1^i, x_2^i, ..., x_n^i]^T$, $B(x^i) \in \mathbb{R}^{n \times m}$, the continuous maps $f^i : \mathbb{R}^n \to \mathbb{R}^n$, $u^i \in \mathbb{R}^m$ is the control input and for $1 \leq k \leq n$ and $1 \leq l \leq m$, $B(x^i) = [b_{kl}]$.

Definition 4 *Given a control-input u^i as protocol, we say that systems in network meet a finite-time average consensus if for any system's state initial conditions, there exists some finite time T_* such that:*

$$\lim_{t \to T_*} \|x^i(t) - \chi(t)\| = 0 \tag{3}$$

for any $i \in \mathcal{I}$, and where $\chi(t) = \dfrac{1}{N} \sum_{j=1}^{N} x^j(t)$ is the average trajectory.

$\chi(t)$ can be interpreted as the instantaneous consent providing that serves the group objectives. χ is time-varying, it can be also considered as the average trajectory of the group, and it is not necessary the average from the multi-system initial conditions. We show that the dynamic of χ depends strongly on the adopted topology of the group.

Subsequently, for the multi-Σ_1 and multi-Σ_2 systems one might analyze the following protocols are given by (4) and (5).
For $i \in \mathcal{I}$, the consensus protocol candidate is given by,

$$u^i = -C(x^i) \sum_{j=1}^{N} a_{ij} \phi_\alpha(x^i - x^j) \tag{4}$$

while the stabilizing input candidate is as

$$u^i = -C(x^i) \sum_{j=1}^{N} a_{ij} (\phi_\alpha(x^i) - \phi_\alpha(x^j)) \tag{5}$$

where the a_{ij} elements are of the $\mathcal{G}$ adjacency matrix, $\alpha \in]0, 1[$, and $\phi_\alpha(.)$ is defined in section 2. The control matrix $C(x^i) \in \mathbb{R}^{m \times n}$ depends on the agent's model, and it will be defined in the following.

As we can see in protocols (4) and (5), the finite-time average consensus is closely related to finite-time stability. The main difference between the two problems is that finite-time average consensus is to make the multi-system converge to an agreement value or trajectory as given by $\chi(t)$ in (3), while the stability of each agent consists to reach an equilibrium. The following assumption gives a conceptual form of $C(x^i)$ with respect to the studied dynamics.

Assumption 5 *The matrix $C(x^i)$ is such that the matrix product $B(x^i)C(x^i)$ is positive semi-definite and diagonalizable.*

Throughout the paper, one denotes by $\tilde{B} = B(x^i)C(x^i)$ where $\tilde{B} = [\tilde{b}_{mk}]_{m,k}$ for $1 \leq m, k \leq n$.

3 Finite-time average consensus

The objective of this section is to solve finite-time average consensus problems of multi-system based on Σ_1 and Σ_2 descriptions. The average value is considered as an agreement function of time but is not necessary function of multi-agent initial conditions. Further, what motivates the analysis is that models given by (2) and (1) cover many autonomous system behaviors affine in the control vector. One may cite, automated highway systems, multi-drone, multi-system of satellites or robots, etc. When we refer to the protocol (4), the interaction topology uses undirected flow information between nodes where each node's vector of states is as a solution of (2) or (1). The following two subsections treat the multi-Σ_1 and multi-Σ_2 finite-time average consensus.

3.1 The multi-Σ_1 finite-time average consensus

For finite-time average consensus of multi-Σ_1 one considers, as interaction topology an undirected fixed graph, an average vector obtained from each Σ_1 vector of states, and the protocol candidate (4). As the matrix B structure is taken identical for each Σ_1 then one might think to networked homogeneous systems. Recall that for a group where each agent is of the form $\dot{x}^i = u^i$, if the interconnection topology is based on an undirected flow, then the average consensus is solved with respect to the average of the agents initial states. Consequently in the following Lemma, for a multi-system based on (1) and (4) we prove that the dynamic of the average is equal to zero, and the agreement remains the average guided by initial states.

Lemma 6 *Given a multi-Σ_1 and a fixed undirected graph, under the protocol (4) the dynamic of the average is equal to zero.*

Proof. Let us calculate the time derivative of $\chi(t)$,

$$\begin{aligned}\dot{\chi}(t) &= \frac{1}{N}\sum_{i=1}^{N}\dot{x}^i(t)\\ &= -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\tilde{B}\phi_\alpha(x^i-x^j)\\ &= -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\tilde{B}\phi_\alpha(x^i-x^j) \qquad (6)\\ &\quad -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\tilde{B}\phi_\alpha(x^i-x^j)\end{aligned}$$

As $a_{ij} = a_{ji}$ (undirected graph) and ϕ_α is an odd function, then it is straightforward to verify that the last equality in (6) leads to $\dot{\chi}(t) = 0$. Consequently, $\chi(t) = cst$. We take $\chi(t) = \chi(0)$ as the average of agents initial states.

■

Proposition 7 *Let $\mathcal{G}$ be connected and a fixed undirected graph, under the protocol (4) and Assumption 5, the multi-Σ_1 achieves a finite-time average consensus in the sense of (3).*

Proof. We introduce $\xi^i(t) = x^i(t) - \chi(t)$. Therefore by Lemma 6, $\dot{\xi}^i(t) = \dot{x}^i(t)$. Let $\xi(t) = (\xi^1, ..., \xi^N)$ and let us take the Lyapunov function candidate

$$V(\xi(t)) = \frac{1}{2}\xi^T\xi = \frac{1}{2}\sum_{i=1}^{N}(\xi^i)^T\xi^i \qquad (7)$$

Due to the fact that $a_{ij} = a_{ji}$ for all $1 \le i, j \le N$, we have

$$\begin{aligned}\dot{V}(\xi(t)) &= \sum_{i=1}^{N}(\xi^i)^T\dot{\xi}^i\\ &= -\sum_{i,j=1}^{N}a_{ij}(\xi^i)^T\tilde{B}\phi_\alpha(\xi^i-\xi^j)\\ &= -\tfrac{1}{2}\sum_{i,j=1}^{N}a_{ij}(\xi^i-\xi^j)^T\tilde{B}\phi_\alpha(\xi^i-\xi^j)\end{aligned}$$

Let $\tilde{B} = PDP^{-1}$, where $D = diag\{0, \mu_2(x^i), ..., \mu_n(x^i)\} \in \mathbb{R}^{n\times n}$, and $\mu_2(x^i), ..., \mu_n(x^i)$ are the eigenvalues of the matrix $\tilde{B}$ given in increasing order such that $\mu_2(x^i) > 0$ for all $x^i \in \mathbb{R}^n$. Therefore,

$$\begin{aligned}\dot{V}(\xi(t)) &\le -\frac{1}{2}\sum_{i,j=1}^{N}a_{ij}\mu_2(x^i)\|\xi^i-\xi^j\|^{\alpha+1}\\ &\le -\frac{1}{2}\sum_{i,j=1}^{N}(a_{ij}\mu_2(x^i))^{\frac{2}{\alpha+1}}\|\xi^i-\xi^j\|^2)^{\frac{\alpha+1}{2}} \qquad (8)\\ &\le -\frac{1}{2}(\sum_{i,j=1}^{N}(a_{ij}\mu_2(x^i))^{\frac{2}{\alpha+1}}\|\xi^i-\xi^j\|^2)^{\frac{\alpha+1}{2}}\end{aligned}$$

Now we consider $\Theta = [\theta_{ij}] \in \mathbb{R}^{n\times n}$ where $\theta_{ij} = (a_{ij}\mu_2(x^i))^{\frac{2}{\alpha+1}}$. Then by Lemma 2 we have,

$$\sum_{i,j=1}^{N}(a_{ij}\mu_2(x^i))^{\frac{2}{\alpha+1}}\|\xi^i-\xi^j\|^2 = 2\xi^T(L(\Theta)\otimes I_n)\xi$$

So,

$$\frac{\xi^T(L(\Theta)\otimes I_n)\xi}{\|\xi\|^2} \ge \lambda_2(L(\Theta)) > 0$$

$L(\Theta)$ can be viewed is the graph Laplacian of the undirected weighted graph $\mathcal{G}(\Theta)$. Therefore, we can rewrite the last inequality (8)

$$\begin{aligned}\dot{V}(\xi(t)) &\le -2^{\frac{\alpha-1}{2}}(\xi^T(L(\Theta)\otimes I_n)\xi)^{\frac{\alpha+1}{2}}\\ &\le -2^{\frac{\alpha-1}{2}}(\frac{\xi^T(L(\Theta)\otimes I_n)\xi}{\|\xi\|^2})^{\frac{\alpha+1}{2}}(V(\xi(t)))^{\frac{\alpha+1}{2}}\\ &\le -2^{\frac{\alpha-1}{2}}\lambda_2^{\frac{2}{\alpha+1}}(V(\xi(t)))^{\frac{\alpha+1}{2}}\end{aligned}$$

As a result, by Lemma 1, V reaches zero at an estimated finite time

$$T_*(\xi(0)) = \frac{V(\xi(0))^{\frac{1-\alpha}{2}}}{2^{\frac{\alpha-3}{2}}\lambda_2^{\frac{2}{\alpha+1}}(1-\alpha)}$$

Thus, the multi-Σ_1 dynamic system with the protocol (4) solve a finite-time average consensus in the sense of (3), and the average is none other than the multi-agent initial state conditions guided by Lemma 6.

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3.2 The multi-Σ_2 finite-time average consensus

The multi-Σ_2 behavior is based on (2) while the consensus protocol candidate is given by (4). Recall that the Σ_2 dynamic as given by (2) is currently present in controlled autonomous systems. However, the drift term can be linear with respect to the system's state vector or taken in its nonlinear form. These two issues will be analyzed in the following with the adequate sufficient conditions for multi-Σ_2 finite-time average consensus. To do, let us first note that f^i in (2) can be different for each dynamic leading to heterogeneous multi-system. At first, the subsequent analysis is build on this form of $f^i(x^i) \triangleq \tilde{A}x^i$ with $\tilde{A}$ is a constant matrix. A controlled dynamic system with linear drift term is given by,

$$\dot{x}^i = \tilde{A}x^i + B(x^i)u^i \tag{9}$$

where $\tilde{A} \in \mathbb{R}^{n\times n}$ with $\tilde{A} = [\tilde{a}_{p,q}]_{1\leq p,q\leq n}$.

Proposition 8 *Let $\mathcal{G}$ be an undirected and connected graph. Under the protocol (4) and Assumption 5 the multi-Σ_2, built from (9), converges toward an average trajectory and leads to a finite-time average consensus in the sense of (3).*

Proof. One introduces $\xi(t) = x^i(t) - \chi(t)$. The goal is to rewrite equation (9) in closed loop depending on ξ^i and to prove that ξ converges to zero in finite time.
Since $a_{ij} = a_{ji}$ and ϕ_α is an odd function, then we have,

$$\begin{aligned}\dot{\chi}(t) &= \frac{1}{N}\sum_{i=1}^{N}(\tilde{A}x^i + Bu^i)\\ &= \frac{1}{N}\sum_{i=1}^{N}\tilde{A}x^i + \frac{1}{N}\sum_{i=1}^{N}Bu^i \\ &= \frac{1}{N}\sum_{i=1}^{N}\tilde{A}x^i\end{aligned} \tag{10}$$

where by Lemma 6 we have $\sum_{i=1}^{N} Bu^i = 0$. At this stage compared to (6), the dynamic of $\chi(t)$ in (10) is different of zero. Then the average considered here is time-varying (average trajectory). This make our average consensus analysis difficult and different from the case of driftless multi-Σ_1.
Now, let us redefine the closed loop dynamic of an agent under (4),

$$\begin{aligned}\dot{\xi}^i &= \tilde{A}\xi^i + B(x^i)u^i\\ &= \tilde{A}\xi^i - \sum_{i,j=1}^{N} a_{ij}\tilde{B}\phi_\alpha(\xi^i - \xi^j)\end{aligned} \tag{11}$$

Using the Lyapunov function (7), and consider the time derivative of $V(\xi)$ along the networked system trajectories (11), we may write

$$\begin{aligned}\dot{V}(\xi(t)) &= \sum_{i=1}^{N}(\xi^i)^T\dot{\xi}^i\\ &= \sum_{i=1}^{N}(\xi^i)^T\tilde{A}\xi^i - \sum_{i,j=1}^{N} a_{ij}(\xi^i)^T\tilde{B}\phi_\alpha(\xi^i-\xi^j)\\ &\leq \|\tilde{A}\|_\infty\sum_{i=1}^{N}\|\xi^i\|^2 - 2^{\frac{\alpha-1}{2}}\lambda_2^{\frac{2}{\alpha+1}}(V(\xi(t)))^{\frac{\alpha+1}{2}}\\ &\leq \|\tilde{A}\|_\infty V(\xi(t)) - 2^{\frac{\alpha-1}{2}}\lambda_2^{\frac{2}{\alpha+1}}(V(\xi(t)))^{\frac{\alpha+1}{2}}\\ &\leq -V(\xi(t))^{\frac{\alpha+1}{2}}[2^{\frac{\alpha-1}{2}}\lambda_2^{\frac{2}{\alpha+1}} - \|\tilde{A}\|_\infty(V(\xi(t)))^{\frac{1-\alpha}{2}}]\end{aligned} \tag{12}$$

where $\|\tilde{A}\|_\infty = \max_{1\leq p\leq n}\sum_{q=1}^{n}|\tilde{a}_{pq}| > 0$. Since $\frac{1-\alpha}{2} > 0$ and V is continuous function which takes 0 at the origin ($\xi \equiv 0$), there exists an open neighborhood $\Omega \subset \mathbb{R}^{Nn}$ of the origin and the last inequality (12) yields to

$$\dot{V}(\xi(t)) \leq -2^{\frac{\alpha-3}{2}}\lambda_2^{\frac{2}{\alpha+1}}V(\xi(t))^{\frac{\alpha+1}{2}} \tag{13}$$

by Lemma 2, V reaches zero at an estimated time

$$T_*(\xi(0)) = \frac{V(\xi(0))^{\frac{1-\alpha}{2}}}{2^{\frac{\alpha-5}{2}}\lambda_2^{\frac{2}{\alpha+1}}(1-\alpha)} \tag{14}$$

Therefore the networked system based on model (9) combined with the protocol (4) leads to a finite-time average consensus where the agreement is an average trajectory defined by the solution from (10). Therefore, the finite-time average consensus is achieved in the sense of (3).

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In the following, we consider that the drift term in (2) is nonlinear which also commonly present in controlled dynamic systems. Moreover, if the networked dynamic systems is homogenous then the f^i structure is identical, otherwise the multi-system is considered as heterogenous. Our main result in multi-Σ_2 is built on the assumption that $f^i(x^i)$ is a convex function.

Proposition 9 *Let $\mathcal{G}$ be a fixed undirected graph and $f^i(x^i)$ is convex. Under the protocol (4) a homogenous/heterogenous multi-Σ_2 based on (2) converges toward an average trajectory and leads to a finite-time average consensus in the sense of (3).*

Proof. One introduces $\xi^i(t) = x^i(t) - \chi(t)$. The goal is to prove that ξ^i converges to zero in finite time. Since $a_{ij} = a_{ji}$ and ϕ_α is an odd function, then we have,

$$\begin{aligned}\dot{\chi}(t) &= \frac{1}{N}\sum_{i=1}^{N}(f^i(x^i) + Bu^i)\\ &= \frac{1}{N}\sum_{i=1}^{N} f^i(x^i) + \frac{1}{N}\sum_{i=1}^{N} Bu^i \qquad (15)\\ &= \frac{1}{N}\sum_{i=1}^{N} f^i(x^i)\end{aligned}$$

Obviously, the average is time-varying (average trajectory). Now as f^i is assumed to be convex then we have

$$f^i(x^i) - \frac{1}{N}\sum_{i=1}^{N} f^i(x^i) \leq f^i(x^i) - f^i(\frac{1}{N}\sum_{i=1}^{N} x^i)$$

Moreover f^i is locally lipschitz function in an open set $\Omega \subset \mathbb{R}^n$ containing ξ. Therefore,

$$\|f^i(x^i) - \frac{1}{N}\sum_{i=1}^{N} f^i(x^i)\| \leq \|f^i(x^i) - f^i(\chi)\| \leq K_1\|\xi^i\|$$

where $K_1 > 0$ is the lipschitz's constant. The dynamic of each system in the group is defined by

$$\begin{aligned}\dot{\xi}^i &= f^i(x^i) - \frac{1}{N}\sum_{i=1}^{N} f^i(x^i) + B(x^i)u^i\\ &= f^i(x^i) - \frac{1}{N}\sum_{i=1}^{N} f^i(x^i) - \sum_{i,j=1}^{N} a_{ij}\tilde{B}\phi_\alpha(\xi^i - \xi^j)\\ &\leq K_1\|\xi^i\| - \sum_{i,j=1}^{N} a_{ij}\tilde{B}\phi_\alpha(\xi^i - \xi^j) \qquad (16)\end{aligned}$$

Now, for convenience the Lyapunov function is given by (7) and the following holds,

$$\begin{aligned}\dot{V}(\xi(t)) &= \sum_{i=1}^{N}(\xi^i)^T\dot{\xi}^i\\ &\leq K_1\sum_{i=1}^{N}\|\xi^i\|^2 - 2^{\frac{\alpha-1}{2}}\lambda_2^{\frac{2}{\alpha+1}}(V(\xi(t)))^{\frac{\alpha+1}{2}}\\ &\leq K_1V(\xi(t)) - 2^{\frac{\alpha-1}{2}}\lambda_2^{\frac{2}{\alpha+1}}(V(\xi(t)))^{\frac{\alpha+1}{2}}\\ &\leq -V(\xi(t))^{\frac{\alpha+1}{2}}[2^{\frac{\alpha-1}{2}}\lambda_2^{\frac{2}{\alpha+1}} - K_1(V(\xi(t)))^{\frac{1-\alpha}{2}}]\\ &\leq -2^{\frac{\alpha-3}{2}}\lambda_2^{\frac{2}{\alpha+1}}V(\xi(t))^{\frac{\alpha+1}{2}} \qquad (17)\end{aligned}$$

At this stage, one concludes that the multi-Σ_2 established from (2) with the protocol (4) lead to a finite-time average consensus. The estimated settling time is as given by (14).

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Note that if the convexity property of f^i is not satisfied, the alternative is to linearize each Σ_2 system and use the same procedure obtained for a multi-system built from (9).

4 The multi-system finite-time stabilization

The finite-time stabilization problem in networked dynamic systems consists to stabilize individually each system's equilibrium state under some connection rules. Then we consider dynamic systems in network with continuous nonlinear decentralized feedback that integrates the graph theory. The following theoretical framework tackles first to the multi-Σ_1 stabilization problem, the results will be extended after that to the analysis of the multi-Σ_2 stabilization problem.

4.1 The multi-Σ_1 finite-time stabilization

The multi-Σ_1 describes the behavior of driftless systems like kinematic of unicycles and attitude of satellites. Further, one considers here that each system is nonlinear and not necessary fully actuated (dimension of the input vector is fewer than the system degree of freedom).

Proposition 10 *For a given fixed undirected graph $\mathcal{G}$, the protocol (5) applied to multi-Σ_1 solves the stabilizing problem in finite time.*

Proof. From (5), let rewrite the matrix form

$$\mathbf{u} = -(L \otimes I_n)(I_N \otimes C(x^i))\phi_\alpha(\mathbf{x}) \qquad (18)$$

with $\mathbf{x} = (x^1, ...x^N)^T$ and $\mathbf{u} = (u^1, ..., u^N)^T$.
The networked systems from (1) under the stabilizing

protocol (18) takes the following form (using the Kronecker product properties [Cremean, 2003])

$$\begin{aligned}\dot{\mathbf{x}} &= (I_N \otimes B(x_i))\mathbf{u} \\ &= -(I_N \otimes B(x_i))(L \otimes I_n)(I_N \otimes C(x^i))\phi_\alpha(\mathbf{x}) \\ &= -(L \otimes \tilde{B})\phi_\alpha(\mathbf{x}) \qquad (19)\end{aligned}$$

It is obvious from (19) that the equilibrium is zero. The goal is to prove that $\mathbf{x}$ reaches this equilibrium in finite time. Taking the Lyapunov function $V : \mathbb{R}^{Nn} \to \mathbb{R}_+$ such that $\forall\ \mathbf{x} \in \mathbb{R}^{Nn}$

$$V(\mathbf{x}) = \frac{1}{1+\alpha}\mathbf{x}^T\phi_\alpha(\mathbf{x}) \qquad (20)$$

which is positive definite with respect to $\mathbf{x}$. Without loss of generality, if we take $z \in \mathbb{R}^p$ with $z = (z_1, ..., z_p)$, we may write $V(z) = \frac{1}{\alpha+1}\sum_{k=1}^{p}|z_k|^{\alpha+1}$, and its time derivative is $\dot{V}(z) = \sum_{k=1}^{p}\varphi_\alpha(z_k)\frac{dz_k}{dt} = \phi_\alpha^T(z)\frac{dz}{dt}$, with φ_α and ϕ_α are as given in notations.
Now, the time derivative of V from (20) along the trajectories of (19) leads to

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \phi_\alpha^T(\mathbf{x})\frac{d\mathbf{x}}{dt} \\ &= -\phi_\alpha^T(\mathbf{x})(L \otimes \tilde{B})\phi_\alpha(\mathbf{x})\end{aligned}$$

Let

$$D(x^i) = \begin{bmatrix} 0_n & & & \\ & \gamma_2(x^i) & & \\ & & \ddots & \\ & & & \gamma_N(x^i) \end{bmatrix}$$

where $0_n = diag\{0, ..., 0\} \in \mathbb{R}^{n\times n}$, and $\forall j = 2, ..., N$ $\gamma_j(x^i) = \lambda_j(L)\varrho_n(x^i)$ with
$\varrho_n(x^i) = diag\{0, \mu_2(x^i), ..., \mu_n(x^i)\} \in \mathbb{R}^{n\times n}$, and where $\mu_2(x^i), ..., \mu_n(x^i)$ are the eigenvalues of the matrix $\tilde{B}$ given in increasing order. $\lambda_j(L)$ denotes the j^{th} eigenvalue of L. Let them be $\lambda_2(L), ..., \lambda_N(L)$ in increasing order. Since $\mathcal{G}$ is connected (by Lemma 2) $\lambda_2(L) > 0$. Therefore, $\forall x^i$ we have $\lambda_2\mu_2(x^i) > 0$.
Further, since $L \otimes \tilde{B} \in \mathbb{R}^{Nn\times Nn}$ is symmetric matrix, then there exist an orthogonal matrix $P \in \mathbb{R}^{Nn\times Nn}$ such that $L \otimes \tilde{B} = P^T D(x^i)P$. Let $\mathbf{z}_\alpha = P\phi_\alpha(\mathbf{x})$, thus

$$\begin{aligned}\dot{V} &= -\mathbf{z}_\alpha^T D\mathbf{z}_\alpha \\ &\le -\lambda_2\mu_1(x^i)\|\mathbf{z}_\alpha\|^2 \\ &\le -\lambda_2\mu_1(x^i)\|\phi_\alpha(\mathbf{x})\|^2 \qquad (21)\end{aligned}$$

with $\lambda_2\mu_1(x^i) = \min_{\mathbf{z}_\alpha \perp \mathbf{1}_{Nn}} \dfrac{\mathbf{z}_\alpha^T D\mathbf{z}_\alpha}{\mathbf{z}_\alpha^T\mathbf{z}_\alpha}$.
Let $k = \min_{x^i\in\mathbb{R}^N} \lambda_2\mu_1(x^i) > 0$ and $\mathbf{x} = \mathbf{1}_N \otimes x^i = (\tilde{x}_1, ..., \tilde{x}_{Nn})^T$, consequently,

$$\begin{aligned}\dot{V} &\le -k\sum_{i=1}^{Nn}|\varphi_\alpha(\tilde{x}_i)|^2 \\ &\le -k\sum_{i=1}^{Nn}|\tilde{x}_i|^{2\alpha} \\ &\le -k(\sum_{i=1}^{Nn}|\tilde{x}_i|^{\alpha+1})^{\frac{2\alpha}{\alpha+1}} \quad \text{(by Lemma 3)} \qquad (22)\end{aligned}$$

which leads to

$$\dot{V} \le -k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}V^{\frac{2\alpha}{\alpha+1}} \qquad (23)$$

Since $0 < \frac{2\alpha}{\alpha+1} < 1$ and $k(\alpha+1)^{\frac{2\alpha}{\alpha+1}} > 0$, by Lemma 1 the above differential equation shows that V reaches zero in finite time

$$T_*(\mathbf{x}(0)) = \frac{(\alpha+1)V(x(0))^{\frac{1-\alpha}{\alpha+1}}}{(1-\alpha)k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}}$$

Therefore, based on (1), the multi-Σ_1 under the protocol (5) reaches zero in finite-time.

■

4.2 The multi-Σ_2 finite-time stabilization

Recall that the multi-Σ_2 system is based on the following dynamic with nonlinear drift terms

$$\Sigma_2 : \quad \dot{x}^i = f^i(x^i) + B(x^i)u^i \qquad (24)$$

where the f^i structure can be taken different for each system. In this case, we are in presence of heterogenous multi-system. We assume at first that

$$\phi_\alpha^T(x^i)f^i(x^i) \le 0 \qquad (25)$$

and we propose the following,

Proposition 11 *Suppose that the inequality (25) is satisfied. For a given fixed undirected and connected graph $\mathcal{G}$, the protocol (5) associated to multi-Σ_2 solves the stabilizing problem in finite time.*

Proof. Let $\mathbf{x} \in \mathbb{R}^{Nn}$ and $f(\mathbf{x}) = (f^1(x^1), ..., f^N(x^N))^T$. Consider the stabilizing protocol (5), from (18) the multi-Σ_2 dynamic becomes,

$$\dot{\mathbf{x}} = f(\mathbf{x}) - (L \otimes \tilde{B})\phi_\alpha(\mathbf{x}) \qquad (26)$$

Using the Lyapunov function (20), its time derivative is as

$$\dot{V}(\mathbf{x}) = \phi_\alpha^T(\mathbf{x})f(\mathbf{x}) - \phi_\alpha^T(\mathbf{x})(L \otimes \tilde{B})\phi_\alpha(\mathbf{x}) \qquad (27)$$

From hypothesis (25) the first term in (27) is negative. The remaining terms in (27) must verify the inequality given by (23). So, we conclude that the origin of (26) is finite-time stable.

■

Remark 12 *In practice condition (25) on the drift term isn't often verified. For this propose this condition can be relaxed by the following proposition.*

Proposition 13 *If f^i is locally Lipshitz function and $f^i(\mathbf{0}_n) = \mathbf{0}_n$, given an undirected and connected graph $\mathcal{G}$, the multi-Σ_2 origin from (24) and (5) is locally finite-time stable.*

Proof. Recall that the time derivative of the Lyapunov candidate function (20)

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \phi_\alpha^T(\mathbf{x})f(\mathbf{x}) - \phi_\alpha^T(\mathbf{x})(L \otimes \tilde{B})\phi_\alpha(\mathbf{x}) \\ &\leq c\|\phi_\alpha^T(\mathbf{x})\mathbf{x}\| - \phi_\alpha^T(\mathbf{x})(L \otimes \tilde{B})\phi_\alpha(\mathbf{x}) \qquad (28)\end{aligned}$$

where $c > 0$ is the Lipshitz's constant.

Let $\mathbf{x} = \mathbf{1}_N \otimes x^i = (\tilde{x}_1, ..., \tilde{x}_{Nn})^T$, consequently from (22), the inequality (28) permits to write

$$\begin{aligned}\dot{V}(\mathbf{x}) &\leq c\sum_{i=1}^{Nn} |\tilde{x}_i|^{\alpha+1} - k(\sum_{i=1}^{Nn} |\tilde{x}_i|^{\alpha+1})^{\frac{2\alpha}{\alpha+1}} \\ &\leq -V^{\frac{2\alpha}{\alpha+1}}[k(\alpha+1)^{\frac{2\alpha}{\alpha+1}} - cV^{\frac{1-\alpha}{1+\alpha}}] \qquad (29)\end{aligned}$$

where $k = \min\limits_{x^i \in \mathbb{R}^N} \lambda_2\mu_1(x^i)$ defined in the proof of Proposition 10. Since $\frac{1-\alpha}{1+\alpha} > 0$ and V is continuous function which takes 0 at the origin, there exists an open neighborhood $\Omega \subset \mathbb{R}^{Nn}$ of the origin that permits to write

$$\dot{V}(\mathbf{x}) \leq -\frac{k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}}{2}[V(\mathbf{x})]^{\frac{2\alpha}{\alpha+1}} \qquad (30)$$

by Lemma 1, V reaches zero at an estimated finite time

$$T_*(\mathbf{x}(0)) = \frac{(\alpha+1)V(\mathbf{x}(0))^{\frac{1-\alpha}{\alpha+1}}}{2(1-\alpha)k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}}$$

Therefore, based on (24) and (5), the multi-Σ_2 origin is finite-time stable.

■

From the proposed stabilizing protocol, we may conclude that the stability of each agent was asserted from the networked behavior of the group. Further, the drift term is not present in the protocol, however along the proofs, this term is tackled by the control and sufficient conditions on this term were introduced to guarantee the multi-system stability. Note that in individual dynamic system stability problem, the drift term must be compensated by the control-input. Here, the stability of each agent is obtained from the stable behavior of the group. This analysis is supported by the following examples.

5 Illustrative examples

In order to validate the above theoretical framework, some examples are presented in simulation and analyzed. The multi-unicycle kinematics is taken in view of the multi-Σ_1 system. Further as multi-Σ_2 examples, we propose to take a multi-second-order dynamics as system with linear drift term and multiple pendulums integrating nonidentical nonlinear drift terms. The cited examples are expected to achieve finite-time average consensus. At the second stage of the given numerical simulations, the networked dynamical systems stability is handled by tests on multi-unicycle. For consensus and stability objectives, the undirected fixed networked topology (binary graph) is shown by Fig.1.

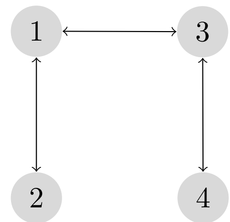

Fig. 1. $\mathcal{G}$ for a system with 4 agents.

5.1 The multi-system finite-time consensus results

Three illustrative examples are considered here where the multi-unicycle that represents the networked systems modeled by (1), a multi-system based on second order dynamic which imply a networked multi-model as in (9), and a multi-pendulum example as in (2). Each associated protocol is deduced from (4).

a) Average consensus in multi-unicycle

Consider N wheeled mobile robots (unicycles) where the i^{th} nonholonomic kinematic model is as:

$$\begin{pmatrix}\dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i\end{pmatrix} = \begin{pmatrix}\cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}u_i \\ w_i\end{pmatrix} \quad i = 1, ..., N \qquad (31)$$

where (x_i, y_i, θ_i) denotes the position and the orientation in a an inertial frame. The inputs u_i and w_i

are the linear and angular velocities, respectively. Let $B = \begin{pmatrix} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \end{pmatrix}$

Based on Proposition 7, the finite-time average consensus problem can be achieved through the following protocol

$$u_i = -\sum_{j=1}^{N} a_{ij}\varphi_\alpha(x_i - x_j)\cos\theta_i - \sum_{j=1}^{N} a_{ij}\varphi_\alpha(y_i - y_j)\sin\theta_i \quad (32)$$

$$w_i = \sum_{j=1}^{N} a_{ij}\varphi_\alpha(x_i - x_j)\sin\theta_i - \sum_{j=1}^{N} a_{ij}\varphi_\alpha(y_i - y_j)\cos\theta_i \quad (33)$$

where φ_α is defined in section 2 and a_{ij} are associated to the graph in Fig.1. The simulation results are limited to $N = 4$ that integrate the following initial conditions

$$(x_1, y_1, \theta_1)(t = 0) = (14, 2, \pi)$$

$$(x_2, y_2, \theta_2)(t = 0) = (-4, 2, -\frac{\pi}{2})$$

$$(x_3, y_3, \theta_3)(t = 0) = (10, 8, \frac{\pi}{2})$$

$$(x_4, y_4, \theta_4)(t = 0) = (-10, -8, 0)$$

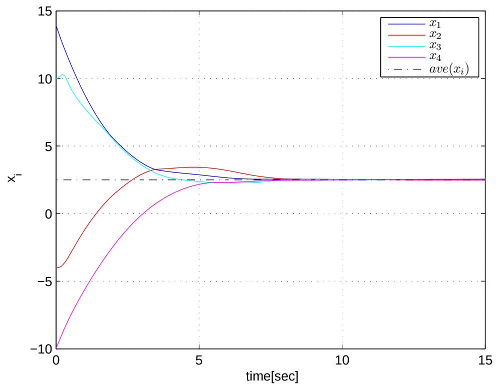

Fig. 2. Average consensus of position x_i for 4 unicycles as multi-Σ_1

The numerical simulations are performed using (31) and protocols (32)-(33). The results of figures Fig. 2-3 evolve according to the developed theoretical results of multi-Σ_1. The common value is also the average of the unicycles initial conditions. The $\|(x_i, y_i) - (ave(x_i(0)), ave(y_i(0)))\|$ converges in finite-time to zero as show in figure Fig.4.

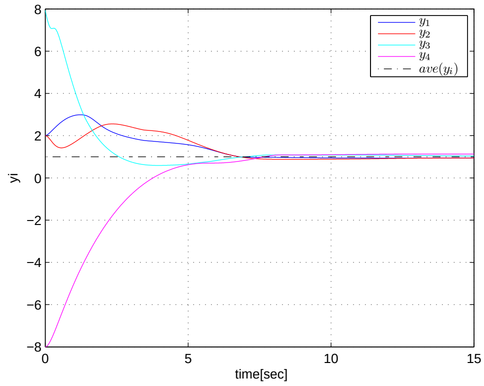

Fig. 3. Average consensus of position y_i for 4 unicycles as multi-Σ_1

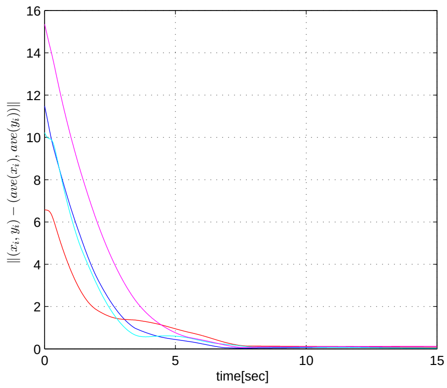

Fig. 4. Convergence of $\|(x_i, y_i) - (ave(x_i), ave(y_i))\|$

b) Average consensus in multi-second-order dynamics

A commonly used example in the literature is an agent with a second-order dynamic (we can see [Wang, 2008])

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= u_i \quad i = 1, ..., N \end{aligned} \quad (34)$$

where $x_i, v_i \in \mathbb{R}$ are the states and $u_i \in \mathbb{R}$ is the control input. The dynamic (34) takes the form given by (9) with $\mathbf{x}^i = \begin{pmatrix} x_i \\ v_i \end{pmatrix}$, $f^i(\mathbf{x}^i) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x}^i$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For the protocol (4) we take $C = (1 \;\; 1)$. From Proposition 8 results, protocols that achieve finite-time average consensus are such that

$$u_i = -\sum_{j=1}^{N} a_{ij}(\varphi_\alpha(x_i - x_j) + \varphi_\alpha(v_i - v_j)) \quad (35)$$

Let us take $N = 4$. The control parameter is taken $\alpha = 0.5$, and each agent initial vector of states is as

$$(x_1, x_2, x_3, x_4)(t = 0) = (5, 10, 1, -5)(meter)$$

and

$$(v_1, v_2, v_3, v_4)(t = 0) = (2, -1, 8, -4)(meter/second)$$

For $i = 1, ..., 4$, x_i (Fig. 5) and v_i (Fig. 6) consent an average trajectory and this was confirmed by the expression (10).

Remark 14 *Other processes can be studied, and where the average is an agreement value of states like a common temperature of sensors where fluctuations of data is important. The energy consumption is also an important factor for stability of electric generators in networks. As example, for a multi-second-order dynamics, the kinetic energies consent an average, and this is shown by figure Fig.7*

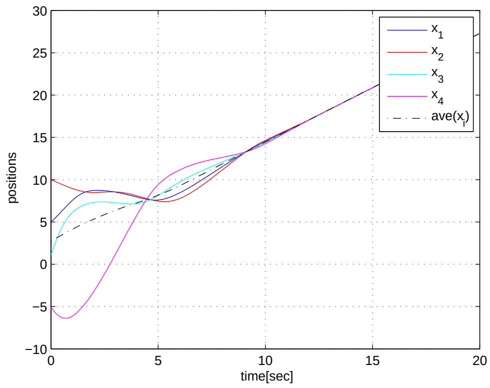

Fig. 5. A reached average trajectory in positions by 4 second-order dynamics

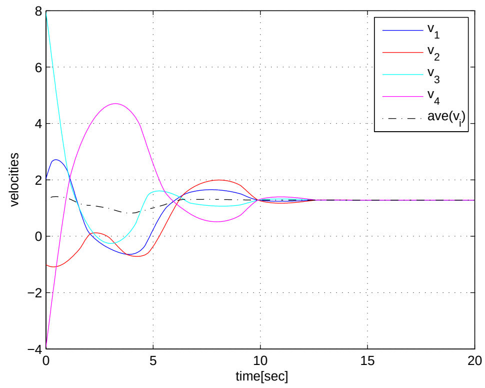

Fig. 6. A reached average trajectory in velocities by 4 second-order dynamics

c) Average consensus in multi-pendulum dynamics

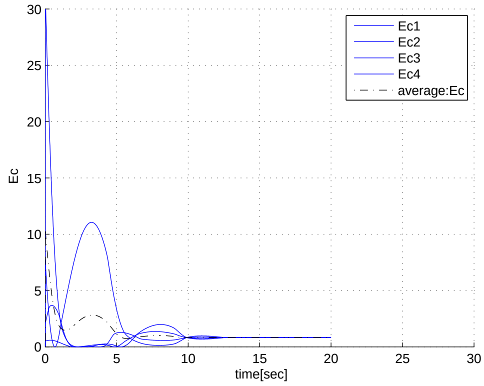

Fig. 7. The average of kinetic energies like consensus for 4 second-order dynamics

Consider a set of N pendulum with the following model

$$\ddot{\theta}_i = -\frac{g}{l_i}\sin(\theta_i) - \frac{\psi_i}{m_i l_i}\dot{\theta}_i + u_i \tag{36}$$

where m_i, g_i, l_i and ψ_i are positive constants. For this system the drift term issues from the first order differential form (see (2)) is

$$f^i(\theta_i, \dot{\theta}_i) = \begin{pmatrix} \dot{\theta}_i \\ -\frac{g}{l_i}\sin(\theta_i) - \frac{\psi_i}{m_i l_i}\dot{\theta}_i \end{pmatrix}$$

we can easily check the convexity condition for the drift term f^i. Following to the subsequent theoretical analysis (see Proposition 9), taking $C = (1\ \ 1)$, a protocol that solves the finite-time average consensus for multi-pendulum is as

$$u_i = -\sum_{j=1}^{N} a_{ij}(\varphi_\alpha(\theta_i - \theta_j) + \varphi_\alpha(\dot{\theta}_i - \dot{\theta}_j)) \tag{37}$$

This set of $N = 4$ pendulums is analyzed. As heterogeneous multi-system, the 4 pendulum parameters aren't similar. Thus, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ and $m_4 = 4$ (Kg). The standard gravity vector is $g = 9.8(\text{m.s}^{-2})$, the lengths $l_i = 1$ (m) and the coefficient $\psi_i = 0.1$ (Kg.m^2.s^{-1}). Initial conditions are such that $\theta_i = (-0.8, 0.4, 1, 2, 1.6)$ (rad) and $\dot{\theta}_i = (0, 0, 0, 0)$ (rad.s^{-1}).
Clearly from figures in Fig. 8-9, the synchronization toward the average trajectory of 4 pendulums in angular positions and velocities are obtained. It is important to note that the average is time-varying and the multi-system of pendulums is heterogeneous with respect to the proposed physical parameters. This confirm the theoretical results of Proposition 9.

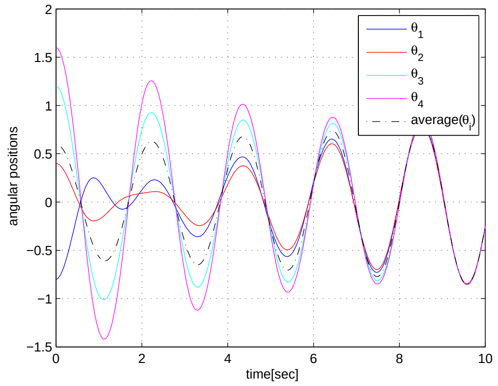

Fig. 8. The time-varying average of angular positions consent by 4 pendulums.

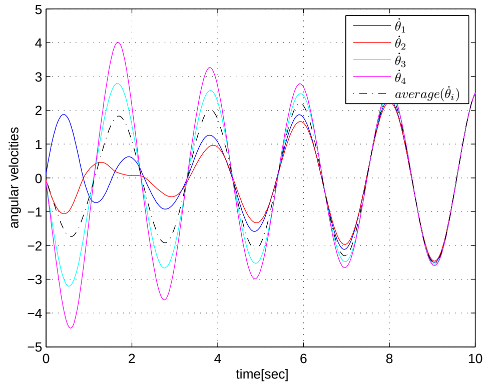

Fig. 9. The time-varying average of angular velocities consent by 4 pendulums.

5.2 *The multi-system finite-time stability results*

We consider a multi-unicycle which represents the networked system modeled by (1) (driftless). The associated protocol is deduced from (5) and the graph is in Fig.1. From Proposition 10, the finite-time stability problem is achieved for the control matrix $C = \begin{pmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \end{pmatrix}$ that leads to the stabilizing control-inputs

$$\begin{aligned} u_i = &-\sum_{j=1}^{N} a_{ij}(\varphi_\alpha(x_i) - \varphi_\alpha(x_j))\cos\theta_i \\ &-\sum_{j=1}^{N} a_{ij}(\varphi_\alpha(y_i) - \varphi_\alpha(y_j))\sin\theta_i \end{aligned} \tag{38}$$

$$\begin{aligned} w_i = &\sum_{j=1}^{N} a_{ij}(\varphi_\alpha(x_i) - \varphi_\alpha(x_j))\sin\theta_i \\ &-\sum_{j=1}^{N} a_{ij}(\varphi_\alpha(y_i) - \varphi_\alpha(y_j))\cos\theta_i \end{aligned} \tag{39}$$

where φ_α is defined in section 2 and a_{ij} are associated to the graph in Fig.1. Taking $N = 4$, the initial conditions are as

$$(x_1, y_1, \theta_1)(t = 0) = (4, 2, \frac{\pi}{4})$$

$$(x_2, y_2, \theta_2)(t = 0) = (12, -10, -\frac{\pi}{2})$$

$$(x_3, y_3, \theta_3)(t = 0) = (10, -8, \frac{2\pi}{3})$$

$$(x_4, y_4, \theta_4)(t = 0) = (-10, -14, \pi)$$

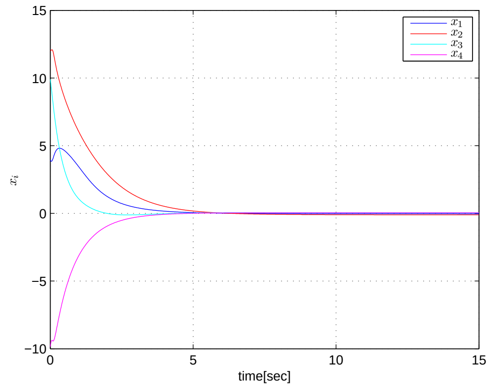

Fig. 10. Finite-time stability of x_i as positions of 4 unicycles

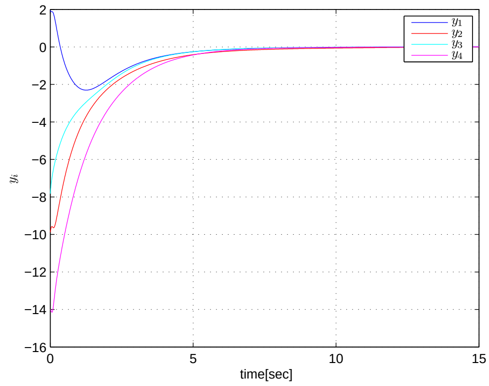

Fig. 11. Finite-time stability of y_i as positions of 4 unicycles

The results of stabilization are sketched in figures Fig.10-11 and the stabilizing protocols are given by figures Fig.12-13 which confirm the stability of each unicycle at the origin with continuous control feedback.

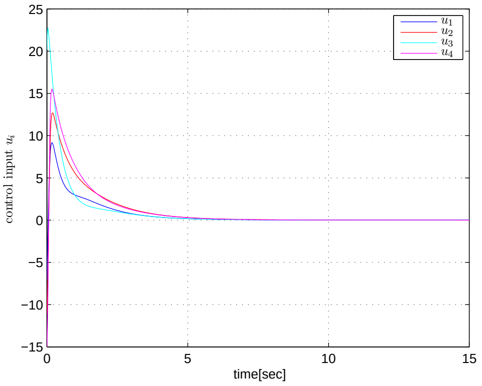

Fig. 12. Stabilizing inputs u_i of 4 unicycles

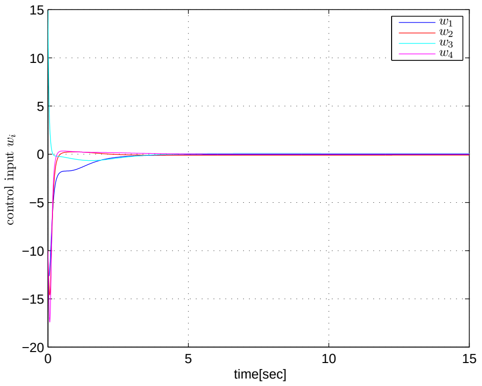

Fig. 13. Stabilizing inputs w_i of 4 unicycles

6 Conclusion

For networked dynamic systems affine in the control vector, two protocols are proposed and theoretically analyzed with respect to two types of nonlinear dynamic models. For a nonlinear driftless multi-system, necessary conditions on the control matrix are derived that assert finite-time average consensus toward a predefined agreement value, obtained from the multi-system initial conditions. However, for multi-system integrating drift terms, sufficient conditions on the drift term are discussed, and when they associated to the protocol solve a finite-time average consensus where as a result an average trajectory is followed by the group. Further, our stability results in networked dynamic systems overcome the individual stability analysis of each system where some obstructions for the agent's stability at the origin occur. It is well known that an unicycle doesn't verify the Brockett's necessary condition and the stabilization at the origin isn't possible with feedbacks that depend only on states. Here, due to the interconnection, the multi-unicycle stability result implies the stability of each unicycle with smooth and bounded control-inputs. The results of the paper can be extended using a directed graph while one may address the problem of consensus and stability for heterogenous systems based on the two fundamental dynamic models.

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