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Finite-time consensus and stability of networked nonlinear systems

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Abstract— **This paper considers the finite-time consensus and stability problems of networked nonlinear systems under an undirected fixed graph. A nonlinear system takes a general form of a controlled first-order differential equation with/without drift term. Sufficient conditions for finite-time consensus and stabilization of networked nonlinear systems are achieved. In multi-system formation, the proposed theoretical approach generalizes the literature results for multi-agent stability and consensus. Heterogenous/homogeneous multi-system formation is a direct application of the given analysis. Some examples are integrated for illustration.**

I. INTRODUCTION

In recent years, the coordination problem of multi-agent systems has received a lot of attention from various scientific searchers due to the diversity of applications in various areas such as mobile robots, air traffic control, scheduling of automated highway systems, unmanned air vehicles, autonomous underwater vehicles, sensor networks and satellites. However, the challenge arising from multi-agent systems is to develop distributed control policies based on local information that enables all agents to reach an agreement on certain quantities of interest, which is known as the consensus problem. The consensus problem was initially used in computer science. In recent years this paradigm has introduced in multi-agent systems witnessed dramatic advances of various distributed strategies that achieve agreements. In [4] the authors proposed a simple but interesting discrete-time model of finite agents all moving in the plane. The proposed model used for the computer animation industry. Each agent's motion is updated using a local rule based on its own state and the states of its neighbors. Jadbabaie et al [5] provided a theoretical explanation of the consensus property of the Vicsek model by using graph theory and nonnegative matrix theory. For this model each agent's set of neighbors changes with time as system evolves. Olfati-Saber and Murray [6] suggested a typical continuous-time model. In this model the concepts of solvability of consensus problems and consensus protocols were first introduced. The authors used a directed graph to model the communication topology among agents and studied three consensus problems, namely, directed networks with fixed topology, directed networks with switching topology, and undirected networks with communication time-delays and fixed topology. Ren and Beard [7] extended the results of Jadbabaie [5] and Olfati-Saber [6] presented mathematically weaker conditions for state consensus under dynamically changing directed interaction topology.

Finite time consensus, which is one of the interesting research problem in consensus, refers to the agreement of a group of agents on a common state in finite time. Finite-time consensus firstly was studied by Cortes [8], where a non-smooth consensus algorithm is proposed. In the same filed [9], and in [12] authors proposed a continuous nonlinear consensus algorithm to guarantee the finite-time stability under an undirected fixed interaction graph. Wang and Xiao in [11] suggest an improvement to the proposed algorithm proposed in [9]. The new algorithm proposed in [11] is able to guarantee finite-time consensus under an undirected switching interaction and a directed fixed interaction graph when each strongly connected component of the topology is detail-balanced. In [14], the authors study finite-time consensus for second order dynamics with inherent nonlinear dynamics under an undirected fixed interaction graph.

Recently, various finite-time stabilizing control laws have been proposed using continuous state feedback and output feedback controllers Bhat et al. [3]. Furthermore, the finite-time control design has been extended to n^{th} order systems with both parametric and dynamic uncertainties [2]. Although the finite-time design is generally more difficult than the asymptotically stabilizing control due to the lack of effective analysis tools. Also, the non-smooth finite-time control synthesis can improve the system behaviors in some aspects like high-speed, control accuracy, and disturbance-rejection. Therefore, it is not surprising that finite-time control ideas have been applied to multi-agent systems with first-order agent dynamics using gradient flow and Lyapunov function [8]-[10].

The present paper was motivate by the lack of methods and results in the literature for finite-time consensus algorithms for networked nonlinear systems. We consider two dynamic models and fixed and undirected graphs. Two types of networked nonlinear systems are studied. The first type describes networked driftless systems, and the second type represents networked drift systems. To solve these problems, we propose nonlinear consensus protocols and modified then for the stabilization objective. Lyapunov function and graph theory are used for the theoretical analysis.

The paper is organized as follows. First, preliminaries and problem formulation are shown in Section II. Then we focus on the finite-time consensus of networked driftless

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systems in section III. Sufficient condition to finite-time consensus of networked drift systems are given in section IV. In section V, we present result for finite-time stabilization of networked driftless/drift nonlinear systems. The paper is ended by concluding remarks.

II. Preliminaries and problem formulation

Throughout this paper, we use $\mathbb{R}$ to denote the set of real number. $\mathbb{R}^n$ is the n-dimensional real vector space and $\|.\|$ denotes the Euclidian norm. $\mathbb{R}^{n\times n}$ is the set of $n \times n$ matrices. $diag\{m_1, m_2, ..., m_n\}$ denotes a $n \times n$ diagonal matrix. $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. The symbol $\otimes$ is the Kronecker product of matrices. We use $sgn(.)$ to denote the signum function. For a scalar x, note that $\varphi_\alpha(x) = sgn(x)|x|^\alpha$. We use $\mathbf{x} = (x_1, ..., x_n)^T$ to denote the vector in $\mathbb{R}^n$. Let $\phi_\alpha(\mathbf{x}) = (\varphi_\alpha(x_1), ..., \varphi_\alpha(x_n))^T$, and $\mathbf{1}_n = (1, ..., 1)^T$. The exponent T is the transpose.

A. Graph theory

In this subsection, we introduce some basic concepts in algebraic graph theory for multi-agent networks. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a directed graph, where $\mathcal{V} = \{1, 2, ..., n\}$ is the set of nodes, node i represents the ith agent, $\mathcal{E}$ is the set of edges, and an edge in $\mathcal{G}$ is denoted by an ordered pair (i, j). $(i, j) \in \mathcal{E}$ if and only if the ith agent can send information to the jth agent directly.

$A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements, where $a_{ij} > 0$ if there is an edge between the ith agent and jth agent and $a_{ij} = 0$ otherwise. Moreover, if $A^T = A$, then $\mathcal{G}$ is also called an undirected graph. In this paper, we will refer to graphs whose weights take values in the set$\{0, 1\}$ as binary and those graphs whose adjacency matrices are symmetric as symmetric. Let $D = diag\{d_1, ..., d_n\} \in \mathbb{R}^{n\times n}$ be a diagonal matrix, where $d_i = \sum_{j=1}^{n} a_{ij}$ for $i = 0, 1, ..., n$. Hence, we define the Laplacian of the weighted graph

$$L = D - A \in \mathbb{R}^{n\times n}.$$

The undirected graph is called connected if there is a path between any two vertices of the graph. Note that time varying network topologies are not considered in this paper.

B. Some useful lemmas

In order to establish our main results, we need to recall the following Lemmas.

Lemma 2.1: [3]. Consider the system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f(0) = 0$, $\mathbf{x} \in \mathbb{R}^n$, there exist a positive definite continuous function $V(\mathbf{x}) : U \subset \mathbb{R}^n \to \mathbb{R}$, real numbers $c > 0$ and $\alpha \in]0, 1[$, and an open neighborhood $U_0 \subset U$ of the origin such that $\dot{V} + c(V(\mathbf{x}))^\alpha \le 0$, $\mathbf{x} \in U_0\backslash\{0\}$. Then $V(\mathbf{x})$ converges to zero in finite time. In addition, the finite settling time T satisfies $T \le \dfrac{V(\mathbf{x}(0))^{1-\alpha}}{c(1-\alpha)}$.

Lemma 2.2: [6]. For a connected undirected graph $\mathcal{G}$, the Laplacian matrix L of $\mathcal{G}$ has the following properties, $\mathbf{x}^T L\mathbf{x} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x_i - x_j)^2$, which implies that L is positive semi-definite. 0 is a simple eigenvalue of L and $\mathbf{1}$ is the associated eigenvector. Assume that the eigenvalues of L are denoted by $0, \lambda_2, ..., \lambda_n$ satisfying $0 \le \lambda_2 \le ... \le \lambda_n$. Then the second smallest eigenvalue satisfies $\lambda_2 > 0$. Furthermore, if $\mathbf{1}^T\mathbf{x} = 0$, then $\mathbf{x}^T L\mathbf{x} \ge \lambda_2 \mathbf{x}^T\mathbf{x}$.

Lemma 2.3: [15]. Let $x_1, x_2, ..., x_n \ge 0$ and $0 < p \le 1$. Then $(\sum_{i=1}^{n} x_i)^p \le \sum_{i=1}^{n} x_i^p \le n^{1-p}(\sum_{i=1}^{n} x_i)^p$.

C. Problem statements

We propose to study the finite-time consensus and stability of two-types of networked nonlinear systems. The first type is given by equation (1) which describes a controlled system without drift. The second type is represented by equation (2) which is clearly a controlled system with drift. One notes that the matrix B for the two models depends on the system's states. Also, in this paper equations (1-2) describe the behavior of an autonomous agent where when we deal with multi-system based on model (1) only, the networked systems is homogenous. However if model (2) and (1) are mixed then the networked systems is heterogenous. Consequently, for networked nonlinear consensus and stability like objectives, models (2) and (1) could generalize the case of multi-agent formation. To the best of our knowledge, consensus and stability problems based on models (1) and (2) have not yet studied.

Consider a group of N high-dimensional agents where each agent's behavior is described by a controlled nonlinear model without drift as given by dynamic (1) and with drift as shown by dynamic (2), $\forall i \in \mathcal{I} = \{1, ..., N\}$

$$\dot{x}_i = B(x_i)u_i \tag{1}$$

and

$$\dot{x}_i = f_i(x_i) + B(x_i)u_i \tag{2}$$

where $x_i \in \mathbb{R}^n$, $B(x_i) \in \mathbb{R}^{n,m}$, the continuous maps $f_i : \mathbb{R}^n \to \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ is the input which depends only on the state of its neighbors.

Definition 2.4: We say that systems in networks based on control-inputs u_i solves a consensus problem in finite time, if for any system's state initial conditions, there exists some finite time T such that $\lim_{t\to T} \|x_i(t) - x_j(t)\| = 0$ for any $i, j \in \mathcal{I}$.

We are now in position to present a consensus protocol, which will be proposed to solve finite-time consensus problems:

For $i \in \mathcal{I}$, let

$$u_i = -C(x_i)\phi_\alpha(\sum_{j=1}^{N} a_{ij}(x_i - x_j)) \tag{3}$$

where $C(x_i) \in \mathbb{R}^{m,n}$, $\alpha \in]0, 1[$ and a_{ij} are the adjacent elements related to $\mathcal{G}$. We assume the following, *Assumption 2.5:* The matrix product $B(x_i)C(x_i)$ is positive semidefinite.

III. Finite-time consensus for networked driftless systems.

We consider networked systems with each system's model is given by (1) and the consensus protocol is as proposed in (3).

Proposition 3.1: Given an undirected and connected graph $\mathcal{G}$, the networked system of form (1) with the protocol (3) lead to a finite-time consensus.

Proof. For $\mathbf{x} = (x_1, ...x_N)^T$ and $\mathbf{u} = (u_1, ..., u_N)^T$, the networked systems is defined by:

$$\dot{\mathbf{x}} = I_N \otimes B(x_i)\mathbf{u} \tag{4}$$

One starts the analysis by an adequate change of variable, for $i \in \mathcal{I}$, let

$$y_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j) \tag{5}$$

consequently, the protocol (3) is rewritten $u_i = -C(x_i)\phi_\alpha(y_i)$, or in compact form $\mathbf{u} = -(I_N \otimes C(x_i))\phi_\alpha(\mathbf{y})$ where $\mathbf{y} = (y_1, ..., y_N)^T$. Therefore, from (5) we have

$$\mathbf{y} = (L \otimes I_n)\mathbf{x} \tag{6}$$

With the given consensus protocol, the dynamic of the networked system (4)-(6) under $\mathbf{u}$ is given by

$$\begin{aligned}\dot{\mathbf{y}} &= (L \otimes I_n)\dot{\mathbf{x}} \\ &= -(L \otimes I_n)(I_N \otimes B(x_i))(I_N \otimes C(x_i))\phi_\alpha(\mathbf{y}) \\ &= -(L \otimes B(x_i)C(x_i))\phi_\alpha(\mathbf{y})\end{aligned} \tag{7}$$

where in the last step we use Kronecker product properties (see [1]). The goal is to prove that $\mathbf{y}$ reaches zero in finite time. Therefore, taking the Lyapunov function candidate $V : \mathbb{R}^{Nn} \to \mathbb{R}_+$ such that $\forall\ \mathbf{y} \in \mathbb{R}^{Nn}$

$$V(\mathbf{y}) = \frac{1}{1+\alpha}\mathbf{y}^T\phi_\alpha(\mathbf{y}) \tag{8}$$

which is positive definite with respect to $\mathbf{y}$, and consider the time derivative of V along the trajectories of (7), we get

$$\begin{aligned}\dot{V}(\mathbf{y}) &= \phi_\alpha^T(\mathbf{y})\frac{d\mathbf{y}}{dt} \\ &= -\phi_\alpha^T(\mathbf{y})(L \otimes B(x_i)C(x_i))\phi_\alpha(\mathbf{y})\end{aligned} \tag{9}$$

Let

$$D(x_i) = \begin{bmatrix} 0_n & & & \\ & \gamma_2(x_i) & & \\ & & \ddots & \\ & & & \gamma_N(x_i) \end{bmatrix}$$

where $0_n = diag\{0, ..., 0\} \in \mathbb{R}^{n\times n}$, and $\forall j = 2, ..., N$ $\gamma_j(x_i) = \lambda_j(L)\varrho_n(x_i)$ with $\varrho_n(x_i) = diag\{0, \mu_2(x_i), ..., \mu_n(x_i)\} \in \mathbb{R}^{n\times n}$, and where $\mu_2(x_i), ..., \mu_n(x_i)$ are the eigenvalues of the matrix $B(x_i)C(x_i)$ given in increasing order. $\lambda_j(L)$ denotes the j^{th} eigenvalue of L. Let them be $\lambda_2(L), ..., \lambda_N(L)$ in increasing order. Since $\mathcal{G}$ is connected (by Lemma 2.2) $\lambda_2(L) > 0$. Therefore, $\forall x_i$ we have $\lambda_2\mu_2(x_i) > 0$.
Further, since $L \otimes B(x_i)C(x_i) \in \mathbb{R}^{Nn\times Nn}$ is symmetric matrix, then there exist an orthogonal matrix $P \in \mathbb{R}^{Nn\times Nn}$ such that $L\otimes B(x_i)C(x_i) = P^T D(x_i)P$. Let $\mathbf{z}_\alpha = P\phi_\alpha(\mathbf{y})$, thus

$$\begin{aligned}\dot{V} &= -\mathbf{z}_\alpha^T D\mathbf{z}_\alpha \\ &\leq -\lambda_2\mu_1(x_i)\|\mathbf{z}_\alpha\|^2 \\ &\leq -\lambda_2\mu_1(x_i)\|\phi_\alpha(\mathbf{y})\|^2\end{aligned} \tag{10}$$

with $\lambda_2\mu_1(x_i) = \min\limits_{\mathbf{z}_\alpha \perp \mathbf{1}_{Nn}} \dfrac{\mathbf{z}_\alpha^T D\mathbf{z}_\alpha}{\mathbf{z}_\alpha^T\mathbf{z}_\alpha}$.
Let $k = \min\limits_{x_i\in\mathbb{R}^N} \lambda_2\mu_1(x_i) > 0$ and $\mathbf{y} = \mathbf{1}_N \otimes y_i = (\tilde{y}_1, ..., \tilde{y}_{Nn})^T$ consequently,

$$\begin{aligned}\dot{V} &\leq -k\sum_{i=1}^{Nn}|\varphi_\alpha(\tilde{y}_i)|^2 \\ &\leq -k\sum_{i=1}^{Nn}|\tilde{y}_i|^{2\alpha} \\ &\leq -k(\sum_{i=1}^{Nn}|\tilde{y}_i|^{\alpha+1})^{\frac{2\alpha}{\alpha+1}} \quad \text{(by Lemma 2.3)}\end{aligned} \tag{11}$$

Then

$$\dot{V} \leq -k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}V^{\frac{2\alpha}{\alpha+1}} \tag{12}$$

Since $0 < \frac{2\alpha}{\alpha+1} < 1$ and $k(\alpha+1)^{\frac{2\alpha}{\alpha+1}} > 0$, and by Lemma 2.3, the above differential equation gives that V reaches zero in finite time $\dfrac{(\alpha+1)V(y(0))^{\frac{1-\alpha}{\alpha+1}}}{(1-\alpha)k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}}$. Therefore the networked systems given by (1) and the protocol (3) lead to a finite-time consensus. This ends the proof.

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Remark 3.2: From inequality (12), if $\alpha = 1$, then the finite-time consensus becomes an asymptotically consensus.

Remark 3.3: The protocol (3) can be applied for the case of a simple dynamic agent ($\dot{x}_i = u_i$) where $B = 1$ and $x_i \in \mathbb{R}$ (see [11]).

Remark 3.4: A simple choice of the matrix C that satisfies Assumption 2.5 is to take $C = B^T$.

A. Multi-unicycle consensus for the rendezvous problem

We propose to study the finite-time consensus of multi-unicycle for the rendezvous problem which is a similar model given by (1). Consider N wheeled mobile robots where the i^{th} nonholonomic kinematic model is as:

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{pmatrix} = \begin{pmatrix} \cos(\theta_i) & 0 \\ \sin(\theta_i) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_i \\ w_i \end{pmatrix} \quad i = 1, ..., N \tag{13}$$

where (x_i, y_i, θ_i) denotes the position and the orientation in a inertial frame. The inputs u_i and w_i are the linear and angular velocities, respectively. Let

$$B = \begin{pmatrix} \cos(\theta_i) & 0 \\ \sin(\theta_i) & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C = B^T$$

Based on Proposition 3.1, the finite-time consensus for the rendezvous problem of networked unicycles can be solved through the following protocol in inputs:

$$\begin{aligned} u_i = & -\varphi_\alpha(\sum_{j=1}^{N} a_{ij}(x_i - x_j)) \cos\theta_i \\ & - \varphi_\alpha(\sum_{j=1}^{N} a_{ij}(y_i - y_j) \sin\theta_i) \end{aligned} \tag{14}$$

$$w_i = -\varphi_\alpha(\sum_{j=1}^{N} a_{ij}(\theta_i - \theta_j)) \tag{15}$$

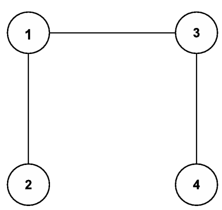

Fig. 1. $\mathcal{G}$ for a system with 4 agents.

For the proposed undirected graph (Fig.1), results for the rendezvous case given by the protocol (14)-(15) are shown in figures (Fig.2-Fig.3). Four unicycles consent on one point in the phase plane. These simulation results imply the following initial conditions $(x_1, y_1, \theta_1)(t = 0) = (4, 2, \frac{\pi}{4})$, $(x_2, y_2, \theta_2)(t = 0) = (2, -1, -\frac{\pi}{2})$, $(x_3, y_3, \theta_3)(t = 0) = (1, 8, \frac{2\pi}{3})$ and $(x_4, y_4, \theta_4)(t = 0) = (-1, -4, \pi)$. $(x_r, y_r, \theta_r)(t = T) = 0$ is the rendezvous common point Fig. 2. The angular positions initially are different and reach a common value in finite time which imply the θ_i consensus (Fig. 3).

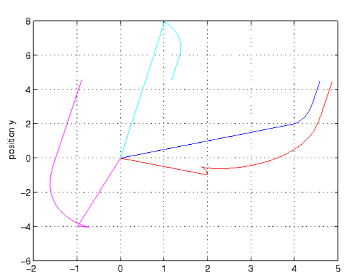

Fig. 2. Phase plot of four unicycles rendezvous under protocols (14)-(15)

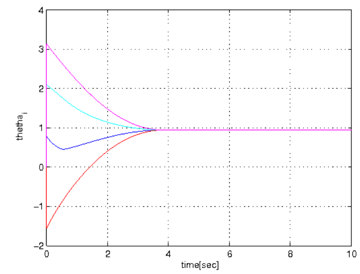

Fig. 3. Angular positions θ_i $(i = 1, ..., 4)$ under protocols (14)-(15)

IV. Finite-time consensus for networked drift systems.

We consider networked systems where each system's model is given by dynamic (2) and the consensus protocol is proposed in (3). Further, we assume that the drift term f_i satisfies the following inequality

$$\| \sum_{j=1}^{N} a_{ij}(f_i(x_i) - f_j(x_j)) \| \leq c \| \sum_{j=1}^{N} a_{ij}(x_i - x_j) \| \tag{16}$$

where c is a positive constant. The networked in homogeneous case implies that the drift term is identical for each system. Here, as shown in (2) the networked systems is taken heterogeneous where we can have different structures of the drift term that satisfies the inequality (16). The goal here is to design u_i in (2) such that $\|x_i(t) - x_j(t)\| \to 0$ in finite time $\forall\ i, j = 1, ..., N$.

Proposition 4.1: Given an undirected and connected graph $\mathcal{G}$, under the inequality (16) the networked systems of form (2) with the protocol (3) lead to a finite-time consensus.

Proof. Using the change of variable given by (5), we have

$$\dot{y}_i = \sum_{j=1}^{N} a_{ij}(f_i(x_i) - f_j(x_j)) + \sum_{j=1}^{N} a_{ij}[B(x_i)u_i - B(x_j)u_j] \tag{17}$$

For $\mathbf{y} = (y_1, ..., y_N)^T$, $f(\mathbf{x}) = (f_1(x_1), ..., f_N(x_N))^T$ and using (6), the networked systems is given by

$$\dot{\mathbf{y}} = (L \otimes I_n) f(\mathbf{x}) - (L \otimes B(x_i) C(x_i)) \phi_\alpha(\mathbf{y}) \tag{18}$$

From inequality (16), we have

$$\|(L \otimes I_n) f(\mathbf{x})\| \leq c \|(L \otimes I_n)\mathbf{x}\| = c\|\mathbf{y}\| \tag{19}$$

Using the Lyapunov function (8), and consider the time derivative of $V(\mathbf{y})$ along trajectories of the networked systems (18), we may write

$$\begin{aligned} \dot{V}(\mathbf{y}) &= \phi_\alpha^T(\mathbf{y})(L \otimes I_n) f(\mathbf{x}) - \phi_\alpha^T(\mathbf{y})(L \otimes B(x_i)C(x_i))\phi_\alpha(\mathbf{y}) \\ &\leq c\|\phi_\alpha^T(\mathbf{y})\mathbf{y}\| - \phi_\alpha^T(\mathbf{y})(L \otimes B(x_i)C(x_i))\phi_\alpha(\mathbf{y}) \end{aligned}$$

Let $\mathbf{y} = \mathbf{1}_N \otimes y_i = (\tilde{y}_1, ..., \tilde{y}_{Nn})^T$, consequently from (11) we get

$$\begin{aligned} \dot{V}(\mathbf{y}) &\leq c \sum_{i=1}^{Nn} |\tilde{y}_i|^{\alpha+1} - k(\sum_{i=1}^{Nn} |\tilde{y}_i|^{\alpha+1})^{\frac{2\alpha}{\alpha+1}} \\ &\leq -V^{\frac{2\alpha}{\alpha+1}}[k(\alpha+1)^{\frac{2\alpha}{\alpha+1}} - cV^{\frac{1-\alpha}{1+\alpha}}] \end{aligned} \tag{20}$$

where $k = \min_{x_i \in \mathbb{R}^N} \lambda_2 \mu_1(x_i)$ defined in the proof of Proposition 3.1. Since $\frac{1-\alpha}{1+\alpha} > 0$ and V is continuous function which takes 0 at the origin, there exists an open neighborhood Ω of the origin that permits to write

$$\dot{V}(\mathbf{y}) \leq -\frac{k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}}{2}[V(\mathbf{y})]^{\frac{2\alpha}{\alpha+1}} \tag{21}$$

by Lemma 2.2, V reaches zero at an estimated finite time

$$T(y(0)) = \frac{(\alpha+1)V(y(0))^{\frac{1-\alpha}{\alpha+1}}}{2(1-\alpha)k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}}$$

Therefore the networked systems based on model (2) and the protocol (3) lead to a finite-time consensus. This ends the proof.

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Remark 4.2: From the proof of Proposition 4.1 if we take $\alpha = 1$, the finite-time consensus becomes an asymptotically consensus.

Example 4.3: Consider a second-order agent dynamics

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= u_i \quad i = 1, ..., N \end{aligned} \tag{22}$$

where $x_i \in \mathbb{R}^n$ denotes the position, $v_i \in \mathbb{R}^n$, and $u_i \in \mathbb{R}^n$ are control inputs. The dynamics (22) takes the form given by (2) with:
$\mathbf{x}_i = \begin{pmatrix} x_i \\ v_i \end{pmatrix}$, $f_i(\mathbf{x}_i) \begin{pmatrix} v_i \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Condition (16) on f_i can be easily verified. Taking $C = \begin{pmatrix} 1 & 1 \end{pmatrix}$ from protocol (3) and Proposition 4.1 we are able to propose the following:

$$u_i = -\varphi_\alpha(\sum_{j=1}^{N} a_{ij}(x_i - x_j)) - \varphi_\alpha(\sum_{j=1}^{N} a_{ij}(v_i - v_j)) \tag{23}$$

For a fixed undirected graph, the double integrator (22) under u_i achieves consensus in positions and velocities. Note that the finite-time consensus for multi-agent networks with second-order agent dynamics as given by (22) was studied by Wang et al. [13]. The consensus protocol proposed here for the double integrator is a direct application of Proposition 4.1, and is different from that given in [13].

Numerical simulation is presented to illustrate consensus of four agents through the graph (Fig. 1). The α control parameter is taken $\alpha = 0.5$, and each agent initial position is $(x_1, x_2, x_3, x_4)(t = 0) = (1, 3, 2, 4)$ (meter) and initial velocity is $(v_1, v_2, v_3, v_4)(t = 0) = (0, -2, 1, 5)$ (meter/second).
Figures in Fig.4 show the effectiveness of the given consensus protocol (23).

V. Finite-time stabilization for networked drift/drifless systems

From Proposition 3.1 proofs, we can distinguish another form of protocol (3) which leads to finite-time stability but do not solve the consensus problem.

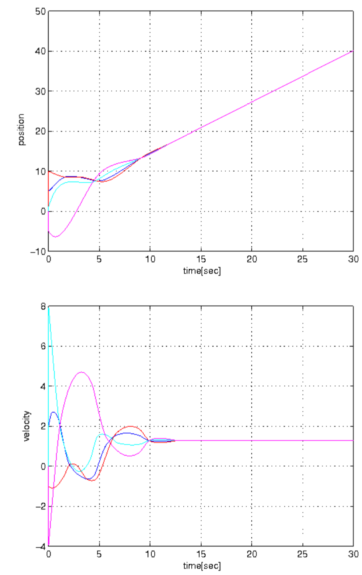

Fig. 4. Position and velocities of the 4 agents

Corollary 5.1: Let

$$u_i = -\sum_{j=1}^{N} a_{ij} C(\phi_\alpha(x_i) - \phi_\alpha(x_j)) \tag{24}$$

Suppose that $\mathcal{G}$ is undirected graph and connected. if the stabilizing protocol (24) is taken for (1), then the resulting networked system states reach zero in finite time.

Proof. The networked systems from (1) under the stabilizing protocol (24) takes the following matrix form

$$\dot{\mathbf{x}} = -[L \otimes B(x_i)C(x_i)]\phi_\alpha(\mathbf{x}) \tag{25}$$

Consider the Lyapunov function (8) with respect to the state $\mathbf{x}$, it is straightforward to prove (steps are similar to Proposition 3.1),

$$\dot{V}(\mathbf{x}) \leq -k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}[V(\mathbf{x})]^{\frac{2\alpha}{\alpha+1}} \tag{26}$$

Therefore, the origin of networked systems based on (1) is finite-time stable, and the estimated settling time is

$$T(x(0)) = \frac{(\alpha+1)V(\mathbf{x}(0))^{\frac{1-\alpha}{\alpha+1}}}{(1-\alpha)k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}}$$

■

Now, the finite-time stability of networked systems modeled by (2) is asserted by the following corollary.

Corollary 5.2: Consider N networked systems where each system is behaved by (2),

$$\dot{x} = f(\mathbf{x}) + (I_N \otimes B(x_i))u \tag{27}$$

assume that,

$$\phi_\alpha^T(x_i) f_i(x_i) \leq 0 \tag{28}$$

Given a connected undirected graph $\mathcal{G}$, under the protocol (24) the origin of (27) is finite-time stable.

Proof. The networked systems is defined by (27) where $\mathbf{x} \in \mathbb{R}^{Nn}$, $u \in \mathbb{R}^{Nm}$ and $f(\mathbf{x}) = (f_1(x_1), ..., f_N(x_N))^T$. Now, let introduce the stabilizing protocol (24) into (27) then the dynamic becomes,

$$\dot{\mathbf{x}} = f(\mathbf{x}) - (L \otimes B(x_i)C(x_i))\phi_\alpha(\mathbf{x}) \tag{29}$$

The time derivative of the Lyapunov function (8) leads to

$$\dot{V}(\mathbf{x}) = \phi_\alpha^T(\mathbf{x})f(\mathbf{x}) - \phi_\alpha^T(\mathbf{x})(L \otimes B(x_i)C(x_i))\phi_\alpha(\mathbf{x}) \tag{30}$$

From hypothesis (28) the first term in equation (30) is negative. The remaining terms in (30) could verified the inequality given by (26). So, we conclude that the origin of (27) is finite-time stable. This ends the proof.

■

A simple and practical example in networked systems with drift terms can be taken from pendulum. Consensus/stabilization for networked pendulums consists to synchronize angular positions and velocities. The asymptotic case was studied by Cremean and Murray in [1].

Example 5.1: Consider a set of pendulum equation in the linear form

$$\ddot{\theta}_i = -\frac{g_i}{l_i}\theta_i - \frac{\psi_i}{m_i l_i}\dot{\theta}_i + u_i \tag{31}$$

where m_i, g_i, l_i and ψ_i are positive constants. For this system, we can easily check condition (28). Then a stabilizing feedback law with respect to the matrix $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$ is given by

$$u_i = -\sum_{j=1}^{N} a_{ij}(\varphi_\alpha(\dot{\theta}_i) - \varphi_\alpha(\dot{\theta}_j)) \tag{32}$$

We conclude that the finite-time stability of the networked pendulums can be asserted by (32). Note that for the asymptotic stability case we can see Cremean's work [1] where $\alpha = 1$.

Remark 5.1: In practice, condition (28) on the drift term isn't often verified. For this propose condition (28) can be relaxed by the following proposition.

Proposition 5.2: If f_i is locally Lipshitz function and $f_i(\mathbf{0}_n) = \mathbf{0}_n$, given an undirected and connected graph $\mathcal{G}$, using (2) the networked systems origin of from (24) is locally finite-time stable.

Proof. Recall that the time derivative of the Lyapunov candidate function (8)

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \phi_\alpha^T(\mathbf{x})f(\mathbf{x}) - \phi_\alpha^T(\mathbf{x})(L \otimes B(x_i)C(x_i))\phi_\alpha(\mathbf{x}) \\ &\leq \chi\|\phi_\alpha^T(\mathbf{x})\mathbf{x}\| - \phi_\alpha^T(\mathbf{x})(L \otimes B(x_i)C(x_i))\phi_\alpha(\mathbf{x})\end{aligned}$$

where $\chi > 0$ is the Lipshitz's constant. Then from proof of Proposition 4.1 the constant c can be identified to χ. Consequently the rest of the proof follows.

VI. CONCLUSIONS

The controlled dynamic model of autonomous systems are presented in this work by two-types of well known nonlinear and continuous first-order differential equations. This has led to controlled system with and without drift. Based on these two types of system's behavior there has been interest to consensus and stabilizing problems of multi-system in networks. Some protocols are proposed and sufficient conditions are achieved to solve finite-time consensus and stabilities of networked systems. The theoretically results of the paper could solve problems of homogenous and heterogenous strategies of formation. As perspective in multi-system formation based on the two given models, problems related to sharing objectives, obstacle avoidance and collision avoidance can follow the same procedure of analysis.

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