# Finite-time consensus of networked nonlinear systems under directed graph

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*Abstract*— **In this paper, we study the finite-time consensus problem of networked nonlinear systems under directed fixed graph. A nonlinear system is considered as a controlled first-order differential equation with/without drift term commonly used to model autonomous systems. For multi-system formation under directed fixed graph, a protocol is proposed to solve consensus problems in finite time. Guided by finite-time stability techniques and the graph theory, our protocol is applied for multi-system in interaction with different nonlinear models. As illustrative examples, under the proposed protocols, a networked unicycles and second-order dynamics achieve consensus in finite time.**

## I. INTRODUCTION

In the past few years, the cooperative control problem for a group of agents is a popular research topic in decentralized control. Robotics provides direct applications of dynamic multi-agent systems. These concepts are asked if you wish to coordinate a team of robots in order to achieve a task. The robots used may evolve in various environments (land, air, water) and are asked to perform tasks such as exploration of a given area, for example the searching for an energy source, the fleet can also be deployed in an area to perform measures or surveillance operations. A team of robots can be used to determine maintenance of fixed stations scattered in a given region. In all these examples, robots must interact with each other to coordinate. Communication tools available (radio, wifi, camera ...) often have a limited scope. The preservation of the connection the group becomes one of the objectives to be met for the task to be accomplished successfully. One possibility to ensure this constraint is moving in training to preserve the geometrical structure of the group. When several agents interact and exchange data, it is often necessary that they can agree on common values (a goal, a place of rendezvous, a distribution of the workload, etc ...). The coherent movement in masses is called consensus. Thus, the problem of consensus plays a central role in study of multi-agent systems. Under the control of a group of mobile agents, it is desirable to obtain coherent and collective movement of agent: displacement close to each other, collision avoidance and a commonly direction. The study of dynamic multi-agent systems makes use of various branches of mathematics. We briefly present those used in this work, starting with the most central: graph theory. Applications of multi-agent dynamic systems are all based on objects that can be similarly abstract: it is a group in which local interactions are generated. This set of interactions can be represented as a network of interactions (which may vary in short time). The mathematical object adapted to model these networks is the graph, the agents are then represented by the nodes of the graph and their interactions through its links. Graph theory, which focuses on the study of these objects, is thus naturally brought into play.

It is important to note that most of consensus problems treated in the literature have been mainly concerned with agent modeled by a first or second order dynamics whose trajectories converge asymptotically to a common value [4], [5], [6].

In some practical situations, it is required that the consensus be reached in a finite time. Consequently, finite-time consensus is more appealing and refers to the agreement of a group of agents on a common state in finite time. . Finite-time consensus firstly was studied by Cortes [7], where a non-smooth consensus algorithm is proposed. In the same field [8] [10] authors propose a continuous nonlinear consensus algorithm to guarantee the finite-time stability under an undirected and fixed graph. Wang and Xiao in [9] suggest an improvement to the algorithm proposed in [8]. The new algorithm proposed in [9] is able to guarantee finite-time consensus under an undirected switching interaction and a directed fixed interaction graph when each strongly connected component of the topology is detail-balanced. In [13], the authors study finite-time consensus for second order dynamics with inherent nonlinear dynamics under an undirected fixed interaction graph. In [11], the authors propose a finite-time consensus and prove the stability of networked nonlinear systems under an undirected fixed graph. The networked multi-system considered in [11] is highly nonlinear system and takes a general form of a controlled first-order differential equation.

Theoretically, consensus based on undirected graph is straightforward as the adjacency matrix is symmetric. However, using a directed graph this leads to a nonsymmetric adjacency matrix, consequently, the consensus study remains a challenge problem. The paper tackles to the finite-time consensus under directed graph of multi-system highly nonlinear with/without drift term in the model. Inspired from finite-time stability results presented in [3], [2] and the graph theory [1], nonlinear consensus protocols are solved

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throughout the paper.

The paper is organized as follows. Some preliminaries results, the problem statement and the finite-time consensus protocol are formulated in section II. In section III one solves a finite-time consensus of multi-system without drift terms. The finite-time consensus of multi-system with drift is detailed in section IV. Finally, illustrative examples are presented in section V.

## II. Preliminaries and problem formulation

### A. Notation

Throughout the paper, we use $\mathbb{R}$ to denote the set of real number. $\mathbb{R}^n$ is the $n$-dimensional real vector space and $\|.\|$ denotes the Euclidian norm. $\mathbb{R}^{n\times n}$ is the set of $n\times n$ matrices. $diag\{m_1, m_2, ..., m_n\}$ denotes a $n\times n$ diagonal matrix. $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. The symbol $\otimes$ is the Kronecker product of matrices. We use $sgn(.)$ to denote the signum function. For a scalar $x$, $\varphi_\alpha(x) = sgn(x)|x|^\alpha$. We use $\mathbf{x} = (x_1, ..., x_n)^T$ to denote the vector in $\mathbb{R}^n$. For $z = (z_1, ..., z_n)$ vector in $\mathbb{R}^n$, the $\delta(z) = [|z_1|, ..., |z_n|]^T$ and $\delta^\gamma(z) = [|z_1|^\gamma, ..., |z_n|^\gamma]^T$ for $\gamma > 0$. Let $\phi_\alpha(\mathbf{x}) = (\varphi_\alpha(x_1), ..., \varphi_\alpha(x_n))^T$, and $\mathbf{1}_n = (1, ..., 1)^T$. The exponent $T$ is the transpose.

### B. Graph theory

In this subsection, we introduce some basic concepts in algebraic graph theory for multi-agent networks. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a directed graph, where $\mathcal{V} = \{1, 2, ..., n\}$ is the set of nodes, node $i$ represents the $i$th agent, $\mathcal{E}$ is the set of edges, and an edge in $\mathcal{G}$ is denoted by an ordered pair $(i, j)$. $(i, j) \in \mathcal{E}$ if and only if the $i$th agent can send information to the $j$th agent directly.
$A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements, where $a_{ij} > 0$ if there is an edge between the $i$th agent and $j$th agent and $a_{ij} = 0$ otherwise. Moreover, if $A^T = A$, then $\mathcal{G}$ is also called an undirected graph. In this paper, we will refer to graphs whose weights take values in the set$\{0, 1\}$ as binary and those graphs whose adjacency matrices are symmetric. Let $D = diag\{d_1, ..., d_n\} \in \mathbb{R}^{n\times n}$ be a diagonal matrix, where $d_i = \sum_{j=1}^{n} a_{ij}$ for $i = 0, 1, ..., n$. Hence, we define the Laplacian of the weighted graph

$$L = D - A \in \mathbb{R}^{n\times n}.$$

The undirected graph is called connected if there is a path between any two vertices of the graph. Directed graph is strongly connected if between every distinct pair $(i, j)$ in $\mathcal{G}$, there is a path that begins at $i$ and ends at $j$.

We say that a directed graph has a spanning tree if a subset of the edges forms a spanning tree (where a spanning tree of $\mathcal{G}$ is a directed tree that is spanning subgraph of $\mathcal{G}$).

Note that time varying network topologies are not considered in this paper.

### C. Some useful lemmas

In order to establish our main results, we need to recall the following Lemmas.

*Lemma 2.1:* [3]. Consider the system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f(0) = 0$, $\mathbf{x} \in \mathbb{R}^n$, there exist a positive definite continuous function $V(\mathbf{x}) : U \subset \mathbb{R}^n \to \mathbb{R}$, real numbers $c > 0$ and $\beta \in ]0, 1[$, and an open neighborhood $U_0 \subset U$ of the origin such that $\dot{V} + c(V(\mathbf{x}))^\beta \leq 0$, $\mathbf{x} \in U_0\backslash\{0\}$. Then $V(\mathbf{x})$ converges to zero in finite time. In addition, the finite settling time $T$ satisfies $T \leq \dfrac{V(\mathbf{x}(0))^{1-\beta}}{c(1-\beta)}$.

*Lemma 2.2:* [5].

(i) If $\mathcal{G}$ has a spanning tree, then eigenvalue 0 is algebraically simple and all other eigenvalues are with positive real part.

(ii) If $\mathcal{G}$ is strongly connected, then there exists a positive column vector $w \in \mathbb{R}^n$ such that $w^T L = 0$

*Lemma 2.3:* [9] Suppose $\mathcal{G}$ is strongly connected, and let $w > 0$ such that $w^T L = 0$. Then $diag(w)L + L^T diag(w)$ is the Laplacien matrix of the undirected weighted graph $\mathcal{G}(diag(w)L+L^T diag(w))$. And therefore it is semi-positive definite, 0 is its algebraically simple eigenvalue and $\mathbf{1}$ is the associated eigenvector.

*Lemma 2.4:* [14]. Let $x_1, x_2, ..., x_n \geq 0$ and $0 < p \leq 1$. Then $(\sum_{i=1}^{n} x_i)^p \leq \sum_{i=1}^{n} x_i^p \leq n^{1-p}(\sum_{i=1}^{n} x_i)^p$.

### D. Problem statements

We propose to study the finite-time consensus and stability of two-types of networked nonlinear systems. The first type is given by equation (1) which describes a controlled system without drift. The second type is represented by equation (2) which is clearly a controlled system with drift. One notes that the matrix $B$ for the two models depends on the system's states. Also, in this paper equations (1-2) describe the behavior of an autonomous agent where when we deal with multi-system based on model (1) only, the networked systems is homogenous. However if model (2) and (1) interact then the networked system is heterogenous. Consequently, for networked nonlinear consensus and stability like objectives, models (2) and (1) could generalize the case of multi-agent formation. To the knowledge of the authors, consensus and stability problems based on models (1) and (2) have not yet been studied.

Consider a group of $N$ high-dimensional agents where each agent's behavior is described by a controlled nonlinear model without drift as given by dynamic (1) and with drift as shown by dynamic (2), $\forall i \in \mathcal{I} = \{1, ..., N\}$

$$\dot{x}^i = B(x^i)u^i \tag{1}$$

and

$$\dot{x}^i = f^i(x^i) + B(x^i)u^i \tag{2}$$

where $x^i \in \mathbb{R}^n$, and for $1 \leq i \leq N$ $x^i = [x_1^i, x_2^i, ..., x_n^i]^T$,

$B(x^i) \in \mathbb{R}^{n\times m}$, the continuous maps $f^i : \mathbb{R}^n \to \mathbb{R}^n$ and $u^i \in \mathbb{R}^m$ is the input which depends only on the state of neighbors.

*Definition 2.5:* We say that systems in network under a control-inputs $u^i$ solve a consensus problem in finite time, if for any system's state initial condition, there exists some finite time $T$ such that $\lim_{t\to T} \|x^i(t) - x^j(t)\| = 0$ for any $i, j \in \mathcal{I}$.

We are now in position to present our main consensus protocol for networked nonlinear systems. The validity of the protocol is detailed in section III and IV.

For $i \in \mathcal{I}$, let

$$u^i = -C(x^i)\phi_\alpha(\sum_{j=1}^{N} a_{ij}(x^i - x^j)) \quad (3)$$

where $C(x^i) \in \mathbb{R}^{m\times n}$, $\alpha \in ]0, 1[$ and $a_{ij}$ are the adjacent elements related to $\mathcal{G}$. We assume the following,

*Assumption 2.6:* The matrix product $B(x^i)C(x^i)$ is positive semidefinite.

*Assumption 2.7:* The drift term $f^i$ in (2) satisfies the following inequality

$$\|\sum_{j=1}^{N} a_{ij}(f^i(x^i) - f^j(x^j))\| \leq \mu\|\sum_{j=1}^{N} a_{ij}(x^i - x^j)\| \quad (4)$$

where $\mu$ is a positive constant.

Throughout the paper, one denotes by $\tilde{B} = B(x^i)C(x^i)$ where $\tilde{B} = [\tilde{b}_{mk}]_{m,k}$ with b$1 \leq m, k \leq n$.

## III. Finite-time consensus for multi-system without drift

We consider a networked system where each system's model is given by (1) and the consensus protocol proposed in (3).

*Proposition 3.1:* If $\mathcal{G}$ has a spanning tree and strongly connected, then the protocol (3) associated to multi-system of type (1) solves a consensus problem in finite time.

**Proof**. For $\mathbf{x} = (x^1, ...x^N)^T$ and $\mathbf{u} = (u^1, ..., u^N)^T$, the networked systems is defined by:

$$\dot{\mathbf{x}} = I_N \otimes B(x^i)\mathbf{u} \quad (5)$$

One starts the analysis by an adequate change of variable, for $i \in \mathcal{I}$, let the vector

$$y^i = \sum_{j=1}^{N} a_{ij}(x^i - x^j) \quad (6)$$

therefore

$$\dot{x}^i = \tilde{B}\phi_\alpha(y^i) \quad (7)$$

consequently, the protocol (3) is rewritten $u^i = -C(x^i)\phi_\alpha(y^i)$, or in compact form $\mathbf{u} = -(I_N \otimes C(x^i))\phi_\alpha(\mathbf{y})$ where $\mathbf{y} = (y_1, ..., y_N)^T$. Therefore, from (6) we have

$$\mathbf{y} = (L \otimes I_n)\mathbf{x} \quad (8)$$

With the given consensus protocol, the dynamic of the networked system (5)-(8) under $\mathbf{u}$ is given by

$$\begin{aligned}\dot{\mathbf{y}} &= (L \otimes I_n)\dot{\mathbf{x}} \\ &= -(L \otimes I_n)(I_N \otimes B(x^i))(I_N \otimes C(x^i))\phi_\alpha(\mathbf{y}) \\ &= -(L \otimes \tilde{B})\phi_\alpha(\mathbf{y}) \quad (9)\end{aligned}$$

where in the last step we use Kronecker product properties (see [1]). The goal is to prove that $\mathbf{y}$ reaches zero in finite time.

As $\mathcal{G}$ is strongly connected there exist a vector $\mathbf{w} = [w^1, w^2, ........, w^N]^T \in \mathbb{R}^{n\times N}$, and for $1 \leq i \leq N$ $w^i = [w^i_1, w^i_2, ..., w^i_n]^T$ such that $\mathbf{w}^T L(A) = 0$ (by lemma2.3)

Therefore, taking the Lyapunov function

$$V(\mathbf{y}) = \frac{1}{1+\alpha}\sum_{i=1}^{N} < w^i, \delta^{1+\alpha}(y^i) > \quad (10)$$

where $< ., . >$ denotes the scalar product.

The Lyapunov function can be rewritten in explicit form

$$V(\mathbf{y}) = \frac{1}{1+\alpha}\sum_{i=1}^{N}\sum_{k=1}^{n} w^i_k |y^i_k|^{1+\alpha}$$

Evaluating $V$ along solution of the transformed vector field (9) and using (6)-(7), we obtain

$$\begin{aligned}\dot{V}(\mathbf{y}) &= \sum_{i=1}^{N}\sum_{k=1}^{n} w^i_k \varphi_\alpha(y^i_k)\frac{dy^i_k}{dt} \\ &= \sum_{i=1}^{N}\sum_{k=1}^{n} w^i_k \varphi_\alpha(y^i_k)(\sum_{j=1}^{N} a_{ij}(\dot{x}^i_k - \dot{x}^j_k)) \quad (11) \\ &= \sum_{i=1}^{N}\sum_{k=1}^{n} w^i_k \varphi_\alpha(y^i_k)(\sum_{j=1}^{N}\sum_{m=1}^{n} a_{ij}[\tilde{b}_{km}\varphi_\alpha(y^i_m) - \tilde{b}_{km}\varphi_\alpha])\end{aligned}$$

writing the last equality in matrix form, it is easy to prove that

$$\dot{V}(\mathbf{y}) = -\phi^T_\alpha(\mathbf{y})(I_N \otimes diag(w))(L \otimes \tilde{B})\phi_\alpha(\mathbf{y})$$

Let $E = \frac{1}{2}((diag(w)L \otimes \tilde{B}) + (L \otimes \tilde{B}diag(w))^T$ consequently we obtain

$$\frac{dV}{dt} = -\phi^T_\alpha(\mathbf{y})E\phi_\alpha(\mathbf{y}) \quad (12)$$

consider

$$\Omega = \{\mathbf{z} \in \mathbb{R}^{n\times N} : \mathbf{z}^T\mathbf{z} = 1 \text{ and } z = \phi_\alpha(\vartheta) \text{ for } \vartheta \perp w\}$$

Note that $\Omega$ is a compact set. And the function $z^TEz$ is continuous in $\Omega$, then the minimum exist and nonzero $\min_{z\in\Omega} z^TEz \neq 0$ (by Lemma 2.2). Moreover, $E$ is the Laplacian matrix of undirected weighted graph $\mathcal{G}(E)$, and it is semi definite positive (Lemma 2.3). We obtain $\min_{z\in\Omega} z^TEz > 0$. Let $K_1 = \min_{z\in\Omega} z^TEz > 0$ as $\frac{\phi_\alpha(\mathbf{y})}{\sqrt{\phi^T_\alpha(\mathbf{y})\phi_\alpha(\mathbf{y})}} \in \Omega$, then

$$\frac{\phi^T_\alpha(\mathbf{y})E\phi_\alpha(\mathbf{y})}{\phi^T_\alpha(\mathbf{y})\phi_\alpha(\mathbf{y})} = \frac{\phi^T_\alpha(\mathbf{y})}{\sqrt{\phi^T_\alpha(\mathbf{y})\phi_\alpha(\mathbf{y})}}E\frac{\phi_\alpha(\mathbf{y})}{\sqrt{\phi^T_\alpha(\mathbf{y})\phi_\alpha(\mathbf{y})}} \geq K_1$$

The goal is to prove that the derivative of $V$ satisfies $\dot{V} \leq -cV^\beta$ (by Lemma 2.1). Therefore, using (12), we obtain

$$\begin{aligned}\dot{V} &= \frac{\phi_\alpha^T(\mathbf{y})E\phi_\alpha(\mathbf{y})}{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})}\frac{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})}{V^\beta}V^\beta \\ &\leq -K_1\frac{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})}{V^\beta}V^\beta\end{aligned}$$

As

$$\begin{aligned}\frac{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})}{V^\beta} &= \frac{\sum_{i=1}^N\sum_{k=1}^n |y_k^i|^{2\alpha}}{(\sum_{i=1}^N\sum_{k=1}^n \frac{w_k^i}{\alpha+1}|y_k^i|^{1+\alpha})^\beta} \\ &\geq \frac{\sum_{i=1}^N\sum_{k=1}^n |y_k^i|^{2\alpha}}{\sum_{i=1}^N\sum_{k=1}^n (\frac{w_k^i}{\alpha+1})^\beta|y_k^i|^{(1+\alpha)\beta}} \quad \text{(Lemma 2.4)}\end{aligned}$$

Therefore, we choose $\beta = \frac{2\alpha}{1+\alpha}$, and let $k_2 = \max_i \max_k (\frac{w_k^i}{\alpha+1})^\beta$. Obviously, $k_2 > 0$. Finally, we can prove that there exists $c = \frac{K_1}{K_2} > 0$ meaning that

$$\dot{V}(\mathbf{y}) \leq -c(V(\mathbf{y}))^{\frac{2\alpha}{1+\alpha}} \quad (13)$$

and thus $V$ will reach zero in finite time $T = \frac{(\alpha+1)}{(1-\alpha)c}V(\mathbf{y}(0))^{\frac{1-\alpha}{\alpha+1}}$ (by Lemma 2.1). Therefore, the protocol 3 solves a finite-time consensus problem. This ends the proof.

■

*Remark 3.2:* From inequality (13), if $\alpha = 1$, then the finite-time consensus becomes an asymptotically consensus.

*Remark 3.3:* The protocol (3) can be applied for the case of a controlled particle ($\dot{x}_i = u_i$) where $B = 1$ and $x_i \in \mathbb{R}$ (see [9]).

*Remark 3.4:* A simple choice of the matrix $C$ that satisfies Assumption 2.6 is to take $C = B^T$.

## IV. Finite-time consensus for multi-system with drift

We consider networked systems where each system's dynamic model is given by (2) and the consensus protocol (3). Various autonomous systems are modeled by (2) and Assumption 2.7 can be easily verified. The goal here is to design $u^i$ in (2) such that $\|x^i(t) - x^j(t)\| \to 0$ in finite time $\forall\ i, j = 1, ..., N$.

*Proposition 4.1:* If the graph $\mathcal{G}$ has a spanning tree and strongly connected and the drift term satisfies the inequality (4), then networked drift system of type (2) with the protocol (3) lead to a finite-time consensus.

**Proof**. Using the change of variable given by (6), we have

$$\dot{y}^i = \sum_{j=1}^N a_{ij}(f^i(x^i) - f^j(x^j)) + \sum_{j=1}^N a_{ij}[B(x^i)u^i - B(x^j)u^j] \quad (14)$$

For $\mathbf{y} = (y^1, ..., y^N)^T$, $f(\mathbf{x}) = (f^1(x^1), ..., f^N(x^N))^T$ and using (8), the networked system is given by

$$\dot{\mathbf{y}} = (L \otimes I_n)f(\mathbf{x}) - (L \otimes \tilde{B})\phi_\alpha(\mathbf{y}) \quad (15)$$

From inequality (4), we have

$$\|(L \otimes I_n)f(\mathbf{x})\| \leq c\|(L \otimes I_n)\mathbf{x}\| = c\|\mathbf{y}\| \quad (16)$$

Using the Lyapunov function (10), and consider the time derivative of $V(\mathbf{y})$ along the networked system trajectories (15), we may write

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^N\sum_{k=1}^n w_k^i\varphi_\alpha(y_k^i)\frac{dy_k^i}{dt} \\ &= \sum_{i=1}^N\sum_{k=1}^n w_k^i\varphi_\alpha(y_k^i)(\sum_{j=1}^N a_{ij}(\dot{x}_k^i - \dot{x}_k^j)) \\ &= \sum_{i=1}^N\sum_{k=1}^n w_k^i\varphi_\alpha(y_k^i)(\sum_{j=1}^N a_{ij}(f_k^i(x^i) - f_k^j(x^j))) + \dot{V}_{/(1)}\end{aligned}$$

where $\dot{V}_{/(1)}$ design the derivative of Lyapunov function with respect the driftless system (1) given in previous section and satisfies the inequality (13). Using the Assumption 2.7 and the equality (11), we obtain

$$\begin{aligned}\dot{V}(\mathbf{y}) &\leq \mu\sum_{i=1}^N\sum_{k=1}^n w_k^i sgn(y_k^i)|y_k^i|^\alpha(\sum_{j=1}^N a_{ij}(x^i - x^j)) - c(V(\mathbf{y}))^{\frac{2\alpha}{1+\alpha}} \\ &\leq \mu\sum_{i=1}^N\sum_{k=1}^n w_k^i|y_k^i|^{\alpha+1} - c(V(\mathbf{y}))^{\frac{2\alpha}{1+\alpha}} \\ &\leq \mu(1+\alpha)V(\mathbf{y}) - c(V(\mathbf{y}))^{\frac{2\alpha}{1+\alpha}} \\ &\leq -V^{\frac{2\alpha}{\alpha+1}}[c - \mu(1+\alpha)V^{\frac{1-\alpha}{1+\alpha}}] \quad (17)\end{aligned}$$

Since $\frac{1-\alpha}{1+\alpha} > 0$ and $V$ is continuous function which takes 0 at the origin ($\mathbf{y} = \mathbf{0}$), there exists an open neighborhood $\Omega$ of the origin that permits to write

$$\dot{V}(\mathbf{y}) \leq -\frac{c}{2}[V(\mathbf{y})]^{\frac{2\alpha}{\alpha+1}} \quad (18)$$

by Lemma 2.2, $V$ reaches zero at an estimated finite time

$$T(y(0)) = \frac{2(\alpha+1)}{c(1-\alpha)}V(y(0))^{\frac{1-\alpha}{\alpha+1}}$$

Therefore the networked system based on model (2) and the protocol (3) lead to a finite-time consensus. This ends the proof.

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*Remark 4.2:* From the proof of Proposition 4.1 if we take $\alpha = 1$, the finite-time consensus becomes an asymptotically consensus.

## V. ILLUSTRATIVE EXAMPLES

Two illustrative examples are considered where the multi-unicycle represents networked system modeled by (1) (driftless) and the multi-system based on second order dynamic which imply networked multi-model of type (2) (with drift). Each associated protocol is deduced from (3) and the results are illustrated by simulations.

### A. Multi-unicycle consensus for the rendezvous problem

Consider $N$ wheeled mobile robots where the $i^{th}$ non-holonomic kinematic model is as:

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_i \\ w_i \end{pmatrix} \quad i = 1, ..., N \quad (19)$$

where $(x_i, y_i, \theta_i)$ denotes the position and the orientation in a inertial frame. The inputs $u_i$ and $w_i$ are the linear and angular velocities, respectively. Let

$$B = \begin{pmatrix} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C = B^T$$

We propose to study the finite-time consensus of multi-unicycle given by (1). Based on Proposition 3.1, the finite-time consensus problem of the networked system's unicycle can achieved through the following protocol as given by (3):

$$\begin{aligned} u_i &= -\varphi_\alpha(\sum_{j=1}^{N} a_{ij}(x_i - x_j)) \cos\theta_i \\ &\quad - \varphi_\alpha(\sum_{j=1}^{N} a_{ij}(y_i - y_j)) \sin\theta_i \quad (20) \\ w_i &= -\varphi_\alpha(\sum_{j=1}^{N} a_{ij}(\theta_i - \theta_j)) \quad (21) \end{aligned}$$

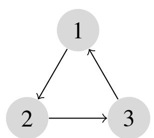


Fig. 1. $\mathcal{G}$ for a system with 3 agents

For the proposed directed graph Fig.1, results for the finite-time consensus case given by the protocol (20)-(21) are shown in figures (Fig.2-Fig.3). Three unicycles consent on one point in the phase plane. These simulation results introduce the following initial conditions $(x_1, y_1, \theta_1)(t = 0) = (-0.5, -0.5, \frac{\pi}{2})$, $(x_2, y_2, \theta_2)(t = 0) = (0.5, -0.5, \frac{\pi}{2})$, $(x_3, y_3, \theta_3)(t = 0) = (0, 0.5, -\frac{2\pi}{2})$. The rendezvous common point is sketched in Fig. 3. The angular positions and the states $(x_i, y_i)$ initially are different and track the same trajectory in finite time which imply the $\theta_i$ consensus (Fig. 2).

### B. Finite-time consensus for multi-agent networks with second-order agent dynamic

Consider a second-order agent dynamics

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= u_i \quad i = 1, ..., N \end{aligned} \quad (22)$$

where $x_i \in \mathbb{R}^n$ denotes the position, $v_i \in \mathbb{R}^n$, and $u_i \in \mathbb{R}^n$ are control inputs. The dynamics (22) takes the form given by (2) with:
$\mathbf{x}_i = \begin{pmatrix} x_i \\ v_i \end{pmatrix}$, $f_i(\mathbf{x}_i) \begin{pmatrix} v_i \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Condition (4) on $f_i$ can be easily verified. Taking $C = \begin{pmatrix} 1 & 1 \end{pmatrix}$, from protocol (3) and Proposition 4.1 we are able to propose the following:

$$u_i = -\varphi_\alpha(\sum_{j=1}^{N} a_{ij}(x_i - x_j)) - \varphi_\alpha(\sum_{j=1}^{N} a_{ij}(v_i - v_j)) \quad (23)$$

For a fixed directed graph, the double integrator (22) under $u_i$ achieves consensus in positions and velocities. Note that the finite-time consensus for multi-agent networks with second-order agent dynamics as given by (22) was studied by Wang et al. in the case of undirected graph [12]. The consensus protocol proposed here for the double integrator is a direct application of Proposition 4.1, and is different from that given in [12].

Numerical simulation is presented to illustrate consensus of three agents through the graph (Fig. **??**). The $\alpha$ control parameter is taken $\alpha = 0.5$, and each agent initial position is $(x_1, x_2, x_3)(t = 0) = (5, 10, 1)$ (meter) and initial velocity is $(v_1, v_2, v_3)(t = 0) = (2, -1, 8)$ (meter/second).
Figures in Fig.4 show the effectiveness of the given consensus protocol (23).

## VI. CONCLUSIONS

The controlled dynamic model of autonomous systems are presented in this work by two-types of well known nonlinear and continuous first-order differential equations. This has led to controlled system with and without drift. Based on these two types of system's behavior there has been interest to consensus problems of multi-system in network. Some protocols are proposed and sufficient conditions are achieved to solve finite-time consensus of networked systems. The theoretically results of the paper could solve problems of homogenous and heterogenous strategies of formation. As perspective in multi-system formation based on the two given models, problems related to sharing objectives, obstacle avoidance and collision avoidance can follow the same procedure of analysis.

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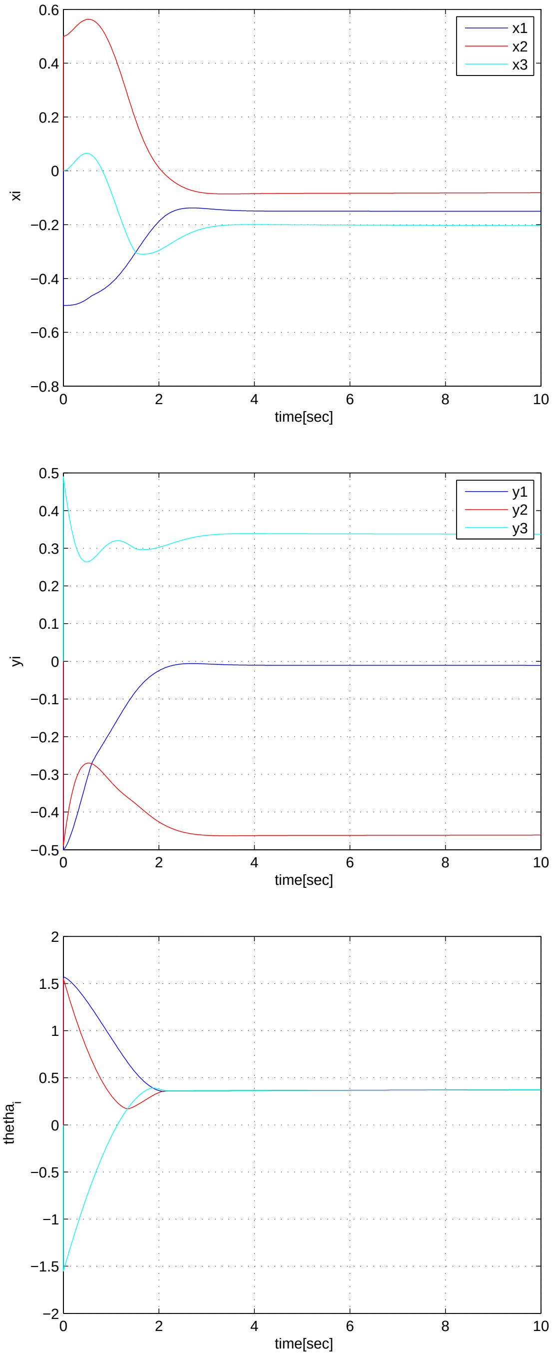


Fig. 2. Consensus results for three unicycles


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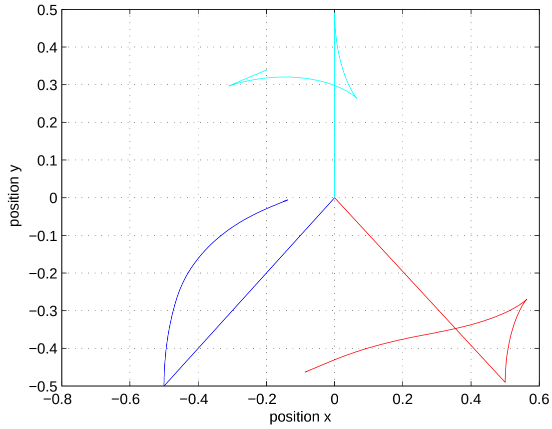


Fig. 3. Phase plot of three unicycles rendezvous

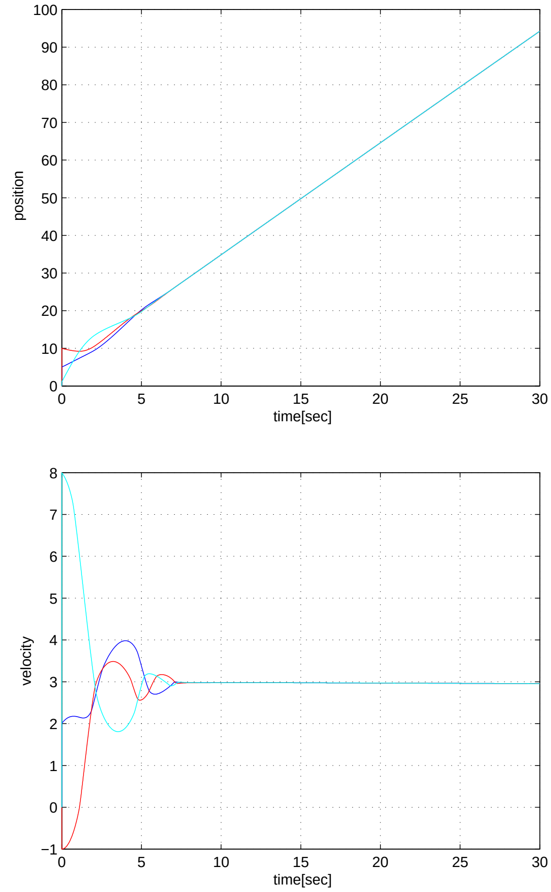


Fig. 4. Position and velocities of three second order dynamics


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