The finite-time stability of perturbed systems

Naim Zoghlami, Lotfi Beji, Rhouma Mlayeh and Azgal Abichou

Abstract— **This paper deals with the finite-time stability of dynamic perturbed systems. The Lyapunov stability case is studied for nonautonomous systems and where the autonomous part is considered as finite-time stable and augmented by a separable function related to time-varying perturbations. As a result, the nonautonomous perturbed system is showed finite-time stable. Sufficient conditions are proposed for finite-time stability of homogeneous and T-periodic systems and where the averaging method has lead to a perturbed average system. The autonomous X4 flyer attitude and position stabilizations are obtained in finite-time. Some simulation results illustrate the proposed stability method.**

I. Introduction

It is well known that finite-time stability is defined for equilibria of continuous but non-Lipschitzian autonomous systems. It involves dynamical systems whose trajectories converge to an equilibrium state in finite time. The problem of this stability theory is motivated by the fact that is more practical concept of stability than is provided by the classical theory. Conditions of stability take the form of existence of Lyapunov-like function whose properties differ significantly from those of classical Lyapunov functions and where difficulties to describe this function persist. In this case, the average techniques are an alternative, because there is no requirement of definiteness on such functions or their derivative. Haimo [2] studied autonomous scalar systems and gives necessary and sufficient conditions for the finite-time stability of the origin. The stability problem of nonautonomous systems was treated by several authors such as Orlov [7] for switched systems, Moulay in [4] gives sufficient conditions for finite-time stability using Lyapunov functions and Haddad [9] provides Lyapunov and converse Lyapunov conditions for finite-time stability. A principal result of finite-time stability for homogeneous nonautonomous systems was obtained by Bhat in [8].

In generally, an autonomous or nonautonomous dynamical system involves perturbations depending in time and the systems states. Then, the dynamical system is presented in the form of two subsystems. One function will describe the autonomous or unperturbed dynamic and the second part regroups all perturbations. Perturbation terms could result from modeling errors, external disturbances. Such a separation leads to a system that belongs to the family of perturbed systems. A general study for asymptotic, exponential stability of perturbed systems was treated in the literature. We can refer to Khalil in [14] for further details. The averaging method is an alternative for T-periodic perturbed systems. Asymptotic stability result for perturbed systems using averaging was proposed by M'Closky in [13], using the same technique. An exponential stability result was developed in [14].

The averaging method is proposed in this paper for locally finite-time stability of nonautonomous dynamic systems, which is in our opinion has not been developed yet. The autonomous average system is considered locally finite-time stable and sufficient conditions are derived for the initial system stability.

Perturbed systems stability in finite-time considers at least that the unperturbed system is finite-time stable and bounding conditions to perturbations are added. We can cite the work of Orlov [7] and Bhat [3]. In this paper, the finite-time stability problem integrates perturbations of the form $h(t,x) = \sum R_i(t)g_i(x)$ where $R_i(t)$ is considered as a separate perturbation. We suppose that this perturbation is simply bounded with respect to time t and is in small neighborhood around zero with respect to the space x. Using the Lyapunov theory and inspired by the work of Moulay [4] and Baht [6], we propose some sufficient conditions leading to the finite-time stability of the origin.

The paper is organized as follows: the second section is devoted to some preliminary mathematical results of finite-time stability and homogeneity. In the third section, we present our main result for a system that is T-periodic and homogeneous using the averaging method. Sufficient conditions for finite-time stability of perturbed autonomous systems including separate perturbations are presented in section IV. Section V details the X4-flyer autonomous aerial vehicle and the finite-time stabilization results for attitudes and positions. Simulation results and conclusions are given in section VI and section VII, respectively.

II. Mathematical preliminaries

In this section we present several preliminary results and definitions which are related to the problem of finite-time stability of nonautonomous systems.

Definition 2.1: [4] Let us consider a nonautonomous dynamic system of the form:

$$\dot{x} = f(t,x) \tag{1}$$

where f is continuous functions in $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$.
The origin is weakly finite-time stable for the system (1) if:

N. Zoghlami is with the IBISC & LIM, University of Evry, IBISC EA 4526 Laboratory, 40 rue du Pelvoux, 91020 Evry, France. Naim.Zoghlami@ibisc.univ-evry.fr

L. Beji is with the IBISC, University of Evry, IBISC EA 4526 Laboratory, 40 rue du Pelvoux, 91020 Evry, France. Lotfi.Beji@ibisc.univ-evry.fr

R. Mlayeh is with the LIM Laboratory, BP743, 2078 La Marsa, Tunisia. rhouma.mlayeh@ipeit.rnu.tn

A. Abichou is with the LIM Laboratory, BP743, 2078 La Marsa, Tunisia. Azgal.Abichou@ept.rnu.tn

(1) the origin is Lyapunov stable for the system (1),
(2) for all $t \in I$, where I is nonempty interval of $\mathbb{R}$, there exists $\delta(t) > 0$, such that if $x \in \mathcal{B}_{\delta(t)}$ then for all $\Phi_t^x \in S(t,x)$:
  i) $\Phi_t^x(\tau)$ is defined for $\tau \geq t$,
  ii) there exists $0 \leq T(\Phi_t^x) < +\infty$ such that $\Phi_t^x(\tau) = 0$ for all $\tau \geq t + T(\Phi_t^x)$.

Let

$$T_0(\Phi_t^x) = \inf\{T(\Phi_t^x) \geq 0 : \Phi_t^x(\tau) = 0 \quad \forall \tau \geq t + T(\Phi_t^x)\}$$

(3) Moreover, if $T_0(t,x) = \sup_{\Phi_t^x \in S(t,x)} T_0(\Phi_t^x) < +\infty$, then the origin is finite-time stable for the system (1).

$T_0(t,x)$ is called the settling time with respect to the initial conditions of the system (1).

Theorem 2.2: [4] Suppose that the origin is an equilibrium point i.e $f(t,0) = 0$ of the system (1).
If there exists a positive definite function **r** such, for $\varepsilon > 0$

$$\int_0^\varepsilon \frac{dz}{\mathbf{r}(z)} < \infty$$

If V is a Lyapunov function continuously differentiable such that

$$\dot{V} \leq -\mathbf{r}(V)$$

then the system (1) is finite-time stable.
The following definitions are useful in the case of a nonautonomous homogeneous system. Further details are in [6],[11] and [12].

Definitions 2.3

- The dilation is considered of the form

$$\Delta_\lambda(x_1, \ldots, x_n) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) \qquad (2)$$

  where $x_1, \ldots, x_n$ are suitable coordinates on $\mathbb{R}^n$ and $r_1, \ldots, r_n$ are positive real numbers. The dilation corresponding to $r_1 = \cdots = r_n = 1$ is the standard dilation in $\mathbb{R}^n$.
- The Euler vector field of the dilation is linear and is given by

$$\nu = r_1 x_1 \partial x_1 + \cdots + r_n x_1 \partial x_n$$

- a function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree l with respect to the dilation (2) if and only if

$$f(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^l V(x_1, \ldots, x_n)$$

- A continuous vector field $X(t,x) = \sum a_i(t,x) \frac{\partial}{\partial x_i}$ on $\mathbb{R} \times \mathbb{R}^n$ is homogeneous of degree $m \leq r_n$ with respect to Δ_λ if a_i is degree $r_i - m$ for $i = 1, ..., m$
- A continuous map from $\mathbb{R}^n$ to $\mathbb{R}$, $x \mapsto \rho(x)$ is called a homogeneous norm with respect to the dilation Δ_λ i.e:
  1) $\rho(x) \geq 0$, $\rho(x) = 0 \Leftrightarrow x = 0$;
  2) $\rho(\Delta_\lambda x) = \lambda \rho(x)\ \forall \lambda > 0$
- The homogeneous norm may always be defined as

$$\rho(x) = |x_1^{\frac{c}{r_1}} + x_2^{\frac{c}{r_2}} + ... x_n^{\frac{c}{r_n}}|^{\frac{1}{c}}$$

  where c is some positive integer evenly divisible by r_i

III. Finite-time stability using averaging

in this section, sufficient conditions are given for finite-time stability of homogeneous and T-periodic systems. Using the averaging method [14] the analysis has lead to a perturbed autonomous system. The averaging result presented here has directly facilitated the stability analysis without using the Lyapunov function.
Let consider the following system :

$$\dot{x} = \varepsilon f(t, x, \varepsilon) \qquad (3)$$

where $x \in \mathbb{R}^n$, $\varepsilon \geq 0$ is a real parameter, f continuous map from $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_+$, T-periodic in t and $f(.,x,.)$ degree $m < 0$ homogeneous with respect to the Euler dilation ν and bounded in x.
The average system is defined by:

$$\dot{y} = \varepsilon f_{av}(y) \qquad (4)$$

with

$$f_{av}(y) = \frac{1}{T} \int_0^T f(\tau, y, 0) d\tau.$$

Note that f_{av} is also of degree $m < 0$ homogeneous with respect to the Euler dilation ν and bounded.
Our result is stated in the following proposition.

Proposition 3.1: Assume that $y = 0$ is finite-time stable fixed point of the associated averaging system (4), then for $\varepsilon > 0$ sufficiently small, the solution $x = 0$ is finite-time stable for the system (3).

□

Proof. The basic problem in the averaging method is to determine in what sense the behavior of the autonomous system (4) approximates the behavior of the nonautonomous system (3). One starts the analysis by an adequate change of variable such that the autonomous system can be represented as a perturbation of the autonomous one. This change of variable is expressed by

$$x = z + \varepsilon u(t, z)$$

where $u(t,x) = \int_0^t [f(\tau, x, 0) - f_{av}(x)] d\tau$.
Differentiating it with respect to time leads to:

$$\dot{x} = \dot{z} + \varepsilon \frac{\partial u}{\partial t}(t,z) + \varepsilon \frac{\partial u}{\partial z}(t,z)\dot{z}$$

therefore, the state equation for z is given by (for further developments we can see [14]).

$$\dot{z} = \varepsilon f_{av}(z) + \varepsilon^2 q(t, z, \varepsilon)$$

The second term represents a perturbation of the average system.
Now let consider a scale change in time $s = \varepsilon t$ that transforms the last equation into

$$\frac{dz}{ds} = f_{av}(z) + \varepsilon q(s/\varepsilon, z, \varepsilon) \qquad (5)$$

where $q(s/\varepsilon, z, \varepsilon)$ is $\varepsilon T-$periodic in s and bounded on $[0, \infty) \times D_0$ for sufficiently small ε.

As f_{av} is homogeneous function of degree $m < 0$ and the origin of the average system (4) is finite-time stable, from Bhat's results [6] there exist a Lyapunov function V, $\mathcal{C}^1$ on $\mathbb{R}^n$ and homogeneous of degree $l > \max\{0, -m\}$ with respect to ν, and there exist $c > 0$, such that $\dot{V} \leq -cV^{\frac{l+m}{l}}$ for the average system (4).

Note that $\sum_{i=1}^{n} \dfrac{\partial V}{\partial x_i} = lV$ and $M = \sup\limits_{(s,z,\varepsilon)} q(s,z,\varepsilon)$

The time derivative of V with respect to system (5)

$$\begin{aligned}\dot{V} \leq & -cV^{\frac{l+m}{l}} + \varepsilon||\nabla V||||q(s/\varepsilon, z, \varepsilon)|| \\ \leq & -V^{\frac{l+m}{l}}[c - lMV^{\frac{-m}{l}}]\end{aligned}$$

Since $\frac{-m}{l} > 0$ and V is continuous function which takes 0 at the origin, there exists an open neighborhood Ω_1 of the origin, and the last inequality now yields

$$\dot{V} \leq -\frac{c}{2} V^{\frac{l+m}{l}}.$$

Then the equilibrium of system (5) is finite-time stable.

It remains to prove that $x = 0$ is finite-time stable for the initial system (3). Using the fact that $x = z + \varepsilon u(t, z)$ with u is bounded in (t, x, ε), then $x(t) - z(s) = O(\varepsilon)$. Hence, $\exists M_1 > 0$ such that $\|x(t) - z(s)\| \leq M_1$

As the equilibrium of (5) is finite-time stable, then there exist T, settling-time function, such that $\lim\limits_{s \to T} z(s) = 0$.

For $\gamma > 0$, there exist $\varepsilon^* = \frac{\gamma}{M} > 0$, and $\exists \beta > 0$ such as $|s - T| < \beta$. Consequently, $\|z(s)\| \leq \gamma - M\varepsilon > 0$ for all $\varepsilon \in]0, \varepsilon^*[$. On the other hand, $\|x(t)\| \leq M\varepsilon + \|z(\varepsilon t\|$, then for $|t - \frac{T}{\varepsilon}| < \frac{\beta}{\varepsilon}$, we have $\|x\| \leq \gamma$. Thus $\lim\limits_{t \to \frac{T}{\varepsilon}} x(t) = 0$ or the finite-time stability of (3).

■

IV. Perturbed autonomous systems

A particular case of perturbed autonomous systems is presented in this section. A time varying disturbance is considered separable from the perturbed term. A perturbed autonomous system is described by:

$$\dot{x} = f(x) + g(t, x) \tag{6}$$

This system can be written in the following form:

$$\dot{x} = f(x) + \sum_{i=1}^{k} R_i(t) g_i(x), x \in \mathbb{R}^n, t \geq 0 \tag{7}$$

where f, g_i $(i = 1, ..., k)$ are continuous functions in $\mathbb{R}^n$ to $\mathbb{R}^n$, R_i $(i = 1, ..., k)$ are continuous functions in $\mathbb{R}_+$.

The objective here is to derive sufficient conditions that guarantee the finite-time stability of system (7).

Proposition 4.1: The equilibrium of the perturbed system (7) is finite-time stable if the following assumptions hold:

1) the equilibrium of the unperturbed system

$$\dot{x} = f(x) \tag{8}$$

is asymptotically stable and homogeneous of degree $m < 0$ with respect to the Euler dilation ν.

2) the function R_i is bounded for all $t \in \mathbb{R}_+$ and $(i = 1, ..., k)$.

3) the function $g_i(x) = O(||x||^{\alpha_i})$ for $(i = 1, ..., k)$ and $x \in \Omega$ where Ω is in neighborhood of the origin, $\alpha_i \in]0, l[$, where $l > \{0, -m\}$ and $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_k$.

□

Proof. From 1) there exist a Lyapunov function V, $\mathcal{C}^1$ on $\mathbb{R}^n$ and homogeneous of degree $l > \max\{0, -m\}$ with respect to ν and there exist $c > 0$, then $\dot{V} \leq -cV^{\frac{l+m}{l}}$ for the unperturbed system (8) [6].

From 2) there exist $M_1 > 0$ such that

$$M_1 = \sup_{t \in \mathbb{R}_+} \sup_{1 \leq i \leq k} R_i(t)$$

and from 3) there exist $M_2 > 0$ such that

$$||g_i(x)|| \leq M_2 ||x||^{\alpha_i}$$

Let $M = \max(M_1, M_2)$.

If V is continuous homogeneous function of degree $l > 0$, then $\frac{\partial V}{\partial x_1} + ... + \frac{\partial V}{\partial x_n} = lV$.

The time derivative of V with respect to system (7)

$$\begin{aligned}\dot{V} \leq & -cV^{\frac{l+m}{l}} + M_1||\nabla V||||g_i(x)|| \\ \leq & -cV^{\frac{l+m}{l}} + lM_2M_1V||x||^{\alpha_i} \\ \leq & -cV^{\frac{l+m}{l}} + lMV||x||^{\alpha_1}\end{aligned} \tag{9}$$

and for a neighborhood Ω_1 of the origin there exist $\gamma > 0$. With

$$\dot{V} \leq -\gamma V^{\frac{l+m}{l}}$$

Then we conclude that if the unperturbed system is finite-time stable, the separate time varying perturbation is bounded, and the vector fields g_i are $O(\|x\|^{\alpha})$ then the equilibrium of the original perturbed system remains stable in finite-time.

■

V. Application: finite-time Stabilization of a X4-flyer

The X4-bidirectional aerial vehicle is minimum in size consisting of four individual electrical fans attached to a rigid bar. Two of them can be oriented by an electric servo-mechanism. This makes the system different of a conventional X4-flyer.

We consider a local reference airframe $\Re_G = \{G, E_1^g, E_2^g, E_3^g\}$ attached to the center of mass G of the areal vehicle. The center of mass is located at the intersection of the two rigid bars, each of them supports two motors. Equipment (controller cards, sensors, etc.) onboard are placed not far from G. The inertial frame is denoted by $\Re_o = \{O, E_x, E_y, E_z\}$ such that the vertical direction E_z is upward. Let the vector $\xi = (x, y, z)$ denote the position of the center of mass of the airframe in the frame $\Re_o$. While the rotation of the rigid body is determined by a rotation $R : \Re_G \to \Re_o$, where $R \in SO(3)$ is an orthogonal rotation matrix. This matrix is defined by the three Euler angles, θ(pitch), ϕ(roll) and ψ(yaw). A sketch of the X4-flyer areal vehicle is given by figure 1 and frames for modeling in figure 2. In the following, we recall only equations due to translations and the attitude yaw dynamic. The reader can refer to [16] for further details in modeling.

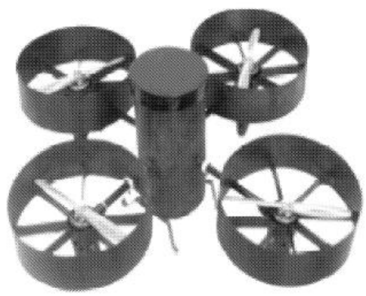

Fig. 1. A general form of the X4-flyer

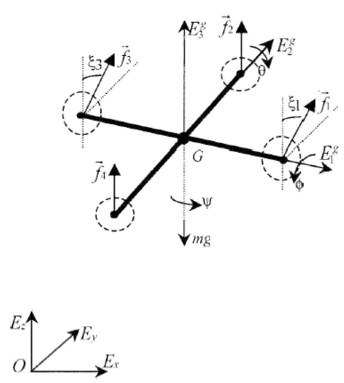

Fig. 2. X4-flyer frames

Before to tackle to the X4-flyer finite-time stabilization problem, in the following, we introduce a general form of a controlled nonlinear system and a finite-time stability result for the double integrator case.

Definition 5.1: [4] The origin is finite-time stabilizable for the controlled system

$$\dot{x} = f(x, u), x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{10}$$

if there exists a control $u \in \mathcal{C}^0(\Omega, \mathbb{R}^m)$ such that

- $u(0) = 0$
- the origin of the system $\dot{x} = f(x, u(x))$ is finite-time stable.

We consider the following reduced model of the X4-flyer where the double integrator dynamics of θ and ϕ can be easily finite-time stable to the origin. A direct input for each attitude variable is asserted through a servomotor and an adequate control can be elaborated.
Our interest concerns the following inertial dynamic model [16]:

$$\begin{aligned} m\ddot{x} &= u\sin(\psi) \\ m\ddot{y} &= u\cos(\psi) \\ m\ddot{z} &= mg - v \\ \ddot{\psi} &= \tau_\psi \end{aligned} \tag{11}$$

where $(x, \dot{x}, y, \dot{y}, z, \dot{z}, \psi, \dot{\psi})^t \in \mathbb{R}^8$, and $(u, v, \tau_\psi) \in \mathbb{R}^3$ is the control vector. By adding integrator, we obtain the following augmented system:

$$\begin{aligned} \dot{x} &= x_1 \\ \dot{x_1} &= \frac{1}{m}\alpha\sin(\beta) \\ \dot{y} &= y_1 \\ \dot{y_1} &= \frac{1}{m}\alpha\cos(\beta) \\ \dot{\alpha} &= u \\ \dot{\beta} &= \psi \end{aligned} \tag{12}$$

Since the finite-time stability of (z, ψ) behavior in the subsystem (13) can be achieved respectively by v and τ_ψ. For the subsystem (13)

$$\begin{aligned} \dot{z} &= z_1 \\ \dot{z_1} &= g - \frac{1}{m}\nu \\ \dot{\psi} &= \omega \\ \dot{\omega} &= \tau_\psi \end{aligned} \tag{13}$$

System (13) takes the forme of a double integrator. The finite-time stabilization of (13) can be achieved through Bhat's results [8], and the following control inputs achieve the finite-time stabilization of the altitude z and the attitude ψ:

$$\begin{aligned} \nu &= m(g + k_1 sign(z)|z|^{\alpha_1} + k_2 sign(z_1)|z_1|^{\alpha_2} \\ \tau_\psi &= -k_1 sign(\psi)|\psi|^{\alpha_1} - k_2 sign(\omega)|\omega|^{\alpha_2} \end{aligned} \tag{14}$$

ensure the finite-time stabilization of (13). $k_1 > 0$, $k_2 > 0$, $\alpha_1 \in (0, 1)$ and $\alpha_2 = \frac{2\alpha_1}{1+\alpha_1}$.

Remark 5.2: The system (12) doesn't verify the necessary conditions of Brockett [10], hence it cannot be stabilized by a static smooth feedback law. A time varying feedback law that overcomes the obstruction due to Brokett and leads to a finite-time stabilization is given in the following proposition.

Proposition 5.3: Let

$$\begin{aligned}\alpha_d =& 2m\rho_\chi \sin(\frac{t}{\varepsilon}) - 2m(\varphi_{1/3}(y) + \varphi_{1/2}(y_1))\\ \beta_d =& -2\frac{\sin(\frac{t}{\varepsilon})}{\rho_\chi}(\varphi_{1/3}(x) + \varphi_{1/2}(x_1))\\ u =& -k_1(\alpha - \alpha_d) - \dot{\alpha_d}\\ \psi =& -k_2(\beta - \beta_d) - \dot{\beta}_d\end{aligned} \quad (15)$$

where $\rho_\chi = |x^2 + x_1^3 + y^2 + y_1^3|^{\frac{1}{6}}$ and the notation $\varphi_a(x) = sign(x)|x|^a$, $x \in \mathbb{R}$, then for any small nonnegative k_1, k_2 and for every $\varepsilon > 0$ sufficiently small, the system (12) is locally stabilizable in finite-time.

□

Proof. The initial system can be rewritten in compact form as following:

$$\dot{X} = F(X, t) \quad (16)$$

where $X = (x, x_1, y, y_1, \alpha, \beta)^t \in \mathbb{R}^6$ and $F(X, t) \in \mathbb{R}^6$ is as

$$F(X, t) = \begin{pmatrix} x_1 \\ \frac{1}{m}\alpha \sin(\beta) \\ y_1 \\ \frac{1}{m}\alpha \cos(\beta) \\ u \\ \psi \end{pmatrix} \quad (17)$$

The associated linearized model is given by:

$$\begin{aligned}\dot{x} =& x_1\\ \dot{x_1} =& \frac{1}{m}\alpha\beta\\ \dot{y} =& y_1\\ \dot{y_1} =& \frac{1}{m}\alpha\\ \dot{\alpha} =& u\\ \dot{\beta} =& \psi\end{aligned} \quad (18)$$

The analysis consists to take part of $u = -k_1 sign(\alpha - \alpha_d)|\alpha - \alpha_d|^{\frac{1}{3}} + \dot{\alpha}_d$ and $\psi = -k_2 sign(\beta - \beta_d)|\beta - \beta_d|^{\frac{1}{3}} + \dot{\beta}_d$ which ensure that $\alpha \to \alpha_d$ and $\beta \to \beta_d$ as finite time. Therefore, in closed loop

$$\begin{aligned}\dot{x} =& x_1\\ \dot{x_1} =& \frac{1}{m}\alpha_d\beta_d\\ \dot{y} =& y_1\\ \dot{y_1} =& \frac{1}{m}\alpha_d\end{aligned} \quad (19)$$

Due the periodic time varying control inputs, the resulting system is also a periodic time varying which can be written in the form:

$$\dot{X} = f(X) + R_1(\frac{t}{\varepsilon})g_1(X) + R_2(\frac{t}{\varepsilon})g_2(X) \quad (20)$$

where

$$f(X) = \begin{pmatrix} x_1 \\ -2(\varphi_{1/3}(x) + \varphi_{1/2}(x_1)) \\ y_1 \\ -2(\varphi_{1/3}(y) + \varphi_{1/2}(y_1)) \end{pmatrix}$$

$$g_1(X) = \begin{pmatrix} 0 \\ \frac{4}{\rho_\chi}(\varphi_{1/3}(x) + \varphi_{1/2}(x_1))(\varphi_{1/3}(y) + \varphi_{1/2}(y_1)) \\ 0 \\ 2\rho_\chi \end{pmatrix}$$

$$g_2(X) = \begin{pmatrix} 0 \\ -4(\varphi_{1/3}(x) + \varphi_{1/2}(x_1)) \\ 0 \\ 0 \end{pmatrix}$$

$R_1(t) = \sin(t)$ and $R_2(t) = \sin^2(t) - \frac{1}{2}$.

The nonautonomous part of the X4-flyer dynamic as given by (12), which is also the average dynamic, is locally finite-time stable. In fact, $f_{av}(X) = f(X)$ is homogeneous of degree (-1) with respect to the dilation $\Delta_\lambda(X, t) = (\lambda^3 x, \lambda^2 x_1, \lambda^3 y, \lambda^2 y_1)$ (for further details see Bhat [6]). As $\dot{X} = h(x, \frac{t}{\varepsilon})$, h regroups the right part of (12). h is continuous, 2π-periodic and of degree (-1) with respect to the dilation, then (12) is locally finite-time stable. This confirm our analysis in the case of a perturbed system using averaging methodology (Proposition 3.1). A similarly proof result can be obtained if we consider Proposition 4.1. All hypothesis in proposition can be easily verified where the nonautonomous part is given by $f(X)$. This ends the proof.

■

VI. Simulation Results

In this section, we present simulation results of the autonomous X4-flyer areal vehicle. Let take $m = 1kg$, $\varepsilon = 0.034$, $k_1 = 0.25$ and $k_2 = 0.25$. The drone was initialized first at position $(x_0, y_0, \psi_0) = (0.1m, 0.5m, -\frac{\pi}{2})$ (figure 3) and second at position $(x_0, y_0, \psi_0) = (-2m, 0.5m, 0rd)$ (figure 4). The two figures confirm the theoretical finite-time stability and also the finite-time stabilization achieved for the application. There will be some real applications in progress for the X4-flyer presented in figure 1. The proposed control inputs for the X4-flyer integrate time-varying trigonometric terms, then in practice, the smoothness of their behaviors must be studied carefully.

VII. Conclusion

The finite-time stability of perturbed dynamic systems was studied using the Lyapunov theory and the average method. For a nonautonomous system, the stability analysis use the average method, consequently, the average system stability turned to literature classical finite-time stability results, obtained in the case of autonomous systems. A second part of the paper treated finite-time stability of perturbed systems but in the case of a separate perturbation. The autonomous part resulting for this separation was considered finite-time stable and sufficient conditions on perturbations were introduced.

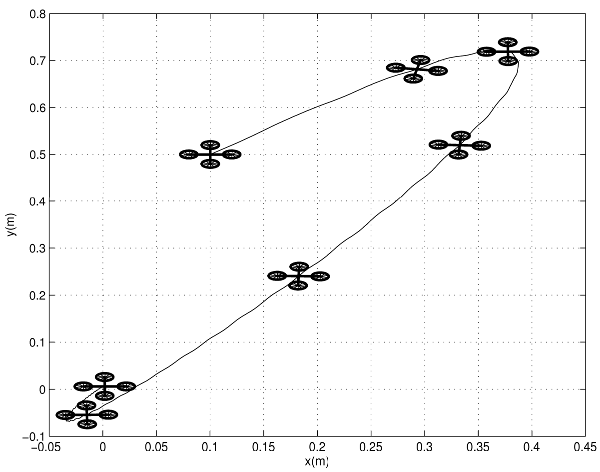

Fig. 3. X4-flyer as finite-time stabilizable at the origin

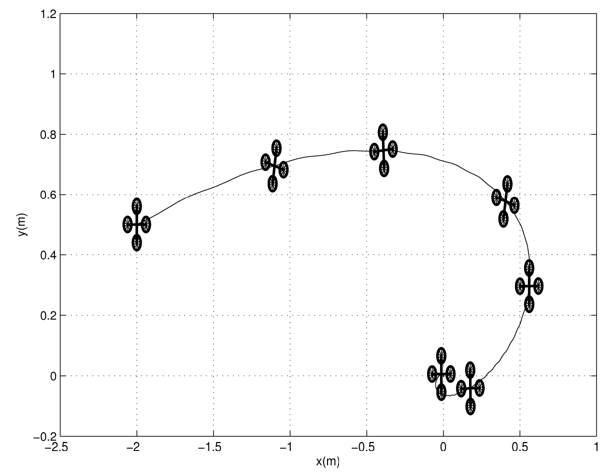

Fig. 4. X4-flyer as finite-time stabilizable at the origin

Conclusions to finite-time stability of perturbed systems in this case were proved with the Lyapunov theory. Finite-time stabilization of X4-flyer attitudes and positions was obtained with the developed theoretical results. Simulations were presented and confirm the proposed stabilizing control-inputs.

REFERENCES

[1] S. P. Bhat and D. S. Bernstein, "Lyapunov Analysis of Finite Time Differential Equations," *Proc. of the ACC 1995 Conference*, Seattle, WA, USA, 1995.

[2] V. T. Haimo, "Finite Time Controllers," *SIAM, J. Control Optim*, 1986, pp. 760–770.

[3] S. P. Bhat and D. S. Bernstein, "Finite Time Stability of Continuous Autonomous Systems," *SIAM J. Control Optim*, vol.38, 2000, pp 751–766.

[4] E. Moulay and W. Perruquetti, "Finite Time Stability of Non Linear Systems", *In IEEE Conference on Decision and Control*, Hawaii, USA, 2003.

[5] S. P. Bhat and D. S. Bernstein, "Continuous Finite-Time Stabilization of the Translational and Rotational Double Integrator," *IEEE Trans. Automat. Control*, vol.43, 1998, pp 678–682.

[6] S. P. Bhat and D. S. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Math. Control Signals Systems*, vol.17, 2005, pp 101–127.

[7] Y. Orlov, "Finite time stability of homogeneous switched systems," *Proc. of the CDC 2003 Conference*, Maui, Hawaii, USA, 2003.

[8] S. P. Bhat and D. S. Bernstein, "Finite-time stability of homogeneous systems," *Proc. of the ACC 1997 Conference*, Albuquerque, NM , USA, 1997.

[9] W. M. Haddad, S. G. Nersesov and D. Liang, "Finite-time stability for time-varying nonlinear dynamical systems," *Proc. of the ACC 2008 Conference*, Seattle, WA, USA, 2008.

[10] R. W. Brokett,"Asymptotic stability and feedback stabilization," *in Differential Geometric Control Theory*, vol.27, 1983, pp.181–191.

[11] M.Kawski, "Homogeneous stabilizing feedback laws," *Control-Theory and Advanced Technology*, vol.6, 1990, pp.497–516.

[12] R. T. M'Closkey and R. M. Murray, "Exponential stabilization of driftless nonlinear control systems using homogeneous feedback," *IEEE Transactions on automatic control*, vol.42, 1997, pp.614–628.

[13] R. T. M'Closkey, "An averaging theorem for time-periodic degree zero homogeneous differential equations," *Systems & Control Letters*, vol.32, 1997, pp.179–183.

[14] H.K.Khalil, *Nonlinear systems (third edition)*, Prentice Hall, 2002.

[15] Y.Hong, "Finite time stabilization and stabilizability of class of controllable systems," *Systems Control Lett*, vol.46, 2002, pp 231–236.

[16] R. Mlayeh, L. Beji and A. Abichou, "B-UAV tracking control integrating planned yaw and longitudinal/lateral inputs," *3rd US-European Workshop and Competition about Micro-Aerial Vehicules, MAV07*, Toulouse, France, 2007.