Regulation control of multi-vehicle formation systems

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Abstract: The paper deals with the stabilising control problem including regulation of multi-vehicle formation systems. By hypothesis, a stabilising control law leading to asymptotic/exponential formation's stabilities toward a target in an undisturbed environment is considered known. However, a novel form of the formation's control input resulting from disturbed agents is emphasised. It is considered as a *regulation control-input* for the system output regulation. The first part of the paper gives conditions on the uniform asymptotic stability of undisturbed systems with/without drift, and the second part shows the control regulation of system paths while avoiding a set of fixed points. Under an adjusted *regulation control-input*, the multi-vehicle formation system's convergence towards a set that surrounds the target is proved using LaSalle's invariance principle. The proposed stabilising control input is smooth among different multi-robot navigation cases.

Keywords: multi-vehicle; obstacle avoidance; stabilisation; regulation control.

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1 Introduction

For control systems in the form $\dot{x} = f(x, u)$ where x is the system's state and u is the control input vector, in the literature, researchers were interested in stabilising of systems and leading to different objectives: asymptotic, uniformly asymptotic, partial, in finite time, etc. To achieve such a result, the used methods involve the following tools: Lyapunov function, LaSalle's invariance principle, Barbalat's Lemma, sliding mode techniques, H_∞ control, etc. In this paper, we are interested in another feature form of the controller, called *regulation control-input*, leading to the output regulation of the closed-loop system in the form $\dot{x} = \mathcal{X}(x, \nu)$. ν is a new function, referred to us as a *regulation control-input* for paths obtained from $\mathcal{X}(x, \nu)$. The subject of output regulation occupies an important role in modern as well as classical theory. The basic problem addressed within output regulation is to design a feedback controller, which internally stabilises a given non-linear system such that the output of the resulting closed-loop system converges to objectives even if external disturbances arise on its trajectory.

In conventional output regulation problem, the statement of the controller is such that the *regulation control-input*, initiated in this work, can not be separated from the original controller. Hence, we treat an output regulation problem but we distinguish it from the conventional one through a *regulation control-input*, which is considered here as a separable form, and it can be designed in a new approach.

The output regulation for linear plants is deeply studied in Saberi's book (Saberi et al., 2000). The non-linear case is treated by Isidori and Byrnes (1990); it is shown that output regulation is locally under some external generator. However, the existence of the controller was proved but no regulator's separate form is given. From a review of the literature, it seems that we can make the following observations for the kinematic controllers of wheeled mobile robots: the stabilisation/tracking controllers do not solve the regulation problem and there are some restrictions given on their differentiability (see McCloskey and Murray, 1997). The regulation problem, related to the kinematic/dynamic of a wheeled mobile robot, is studied by Dixon et al. (2000), and where explicit sinusoidal terms with a tunable frequency are added to the feedback controller. The added sinusoidal term by Dixon et al. (2000) has no physical meaning. The receding horizon method is proposed by Gu and Hu (2006) where two terms in the tracking controller are switched. Generally, the output regulation is solved with motion planning methods associated with kinematic/dynamic models or switching behaviours of the control-input. Note that such a disturbance for mobile robot navigation is caused by obstacles.

The output regulation problem of mobile robots in groups was treated through several strategies. One cites the attractive and repulsive artificial potential functions, which are the mostly used in regulation control. It allows avoiding obstacles and the non-occurrence of collisions between robots (Rimon and Kditschek, 1992; Ge and Cui, 2000a,b; Leonard and Fiorelli, 2009; El Kamel et al., 2009; Kowalczyk et al., 2009; Dong et al., 2006; Essghaier et al., 2011). The null-space-based behavioural approach was proposed by Arrichiello (2006). This method leads to a coordinating behavioural response for conflict resolution including obstacle avoidance, and where the *obstacle-avoidance-task* output generates the appropriate velocity displacement. In term of smoothness of the obtained velocities in Arrichiello (2006), it seems to be a competitive approach.

The objective of the paper is to solve the stabilising control problem including the regulation with respect to some known obstacle's positions with the guarantee of a smooth control and without any motion planning. First, general theoretical results are generalised for systems with and without drift terms. As a result, the obtained control approach ensures the system stability around a desired position and the repulsion of the latter over sets that materialise obstacles. Any form of application will be concerned by our theoretical results including terrestrial and aerial vehicle navigation in an approximatively known environment. Similarly, once planned, the proposed *regulation control-input* may include inter-agent communications or other shared resources. In terms of application, we considered the model of a unicycle-type wheeled vehicle, regardless of orientation (non-holonomic case) and where the environment contains obstacles. Contrary to results presented in the literature, which are based on a switching control strategy for avoidance, our stabilising controller is smooth and continuous over the navigation's space.

The paper is oriented into 8 sections. The first section shows a motivating example of the unicycle-like model, and defines the *regulation control-input* form. Section 3 and 4 treat general cases of uniform asymptotic stability regulation for systems with/without drift. In avoiding some of the sets while ensuring the agent's stability, our main *regulation control-input* is introduced in Section 5. The multi-vehicle formation and the form of the regulation control input are detailed in Section 6. An application to the rendezvous problem is detailed in Section 7. Section 8 concludes the paper with comments.

2 A motivating example

Let us consider the unicycle-like model, which is given by the following equivalent system (see Pomet, 1992):

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 \end{aligned} \tag{1}$$

x_1, x_2 and x_3 are the variable states representing the linear and angular velocities, respectively. u_1 and u_2 are obtained through some preliminary equivalent feedback and denote the two unicycle control inputs. Also, it is an underactuated system without drift. Let us consider system (1) in the form $\dot{q} = P(q)u$, $q = (x_1, x_2, x_3)^T$ and $u = (u_1, u_2)^T$, T denotes the transpose and

$$P(q) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x_1 \end{pmatrix} \tag{2}$$

The unicycle system can be uniformly asymptotically stabilisable according to Pomet (1992) by applying a time-varying feedback law in the form $u_a(q,t) = (u_1(q,t), u_2(q,t))$, with

$$\begin{aligned} u_1(q,t) &= x_2 \sin t - (x_1 + x_3 \cos t) \\ u_2(q,t) &= -(x_1 + x_2 \cos t) x_1 \cos t - (x_1 x_2 + x_3) \end{aligned} \tag{3}$$

Under Pomet's control law (3), the equilibrium asymptotic stability of the non-autonomous system is asserted involving the following Lyapunov function $V(t,q)$:

$$V(t,q) = \frac{1}{2}(x1 + x_2 \cos t)^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 \tag{4}$$

To illustrate the main idea developed in the paper, let us note that the trajectories, as solutions of the controlled system, do not take into account restriction caused by the environment. Such a constraint can be evolved by obstacles, generally belonging to the robot navigation space. As we look to preserve the system stability, any additional input could just modify solutions in the presence of a disturbance. Hence, this additional term will be called *regulation control input*.

Let us compute the gradient of the Lyapunov function (4) with respect to the unicycle variable states

$$\begin{aligned} \partial V/\partial x_1 &= x_1 + x_2 \cos t \\ \partial V/\partial x_2 &= x_1 \cos t + x_2(1 + \cos^2 t) \\ \partial V/\partial x_3 &= x_3 \end{aligned} \tag{5}$$

For any given scalar function $\nu(q,t) : \mathbb{R}^n \to \mathbb{R}$, the following *regulation control input* added to (3) will preserve the system stability at the origin,

$$\nu(q,t) \begin{pmatrix} -x_2(1 + \cos^2 t + x_3) - x_1 \\ x_1 + x_2 \cos t \end{pmatrix} \tag{6}$$

It is trivial to verify that the vector in brackets is orthogonal ($\perp$) to $(\partial V/\partial q)^T P(q)$. This last is given by

$$(\partial V/\partial q)^T P(q) = \begin{pmatrix} x_1 + x_2 \cos t \\ x_2(1 + \cos^2 t + x_3) + x_1 \end{pmatrix} \tag{7}$$

Hence, the scalar product of these two vectors is equal to zero. One expects to preserve the system stability at the origin (details are given in the following section).

We conclude the example by presenting the final form with the unchanged Lyapunov function (4), associated to the unicycle control stability and regulation at the origin

$$\begin{aligned} \widetilde{u}_1(q,t) &= u_1(q,t) + \nu(q,t)(-x_2(1+\cos^2 t + x_3) - x_1) \\ \widetilde{u}_2(q,t) &= u_2(q,t) + \nu(q,t)(x_1 + x_2 \cos t) \end{aligned} \tag{8}$$

The added input ν, considered as a regulation function, plays an important role in the regulation process.

3 Regulation control input for driftless systems

Driftless systems are linear in control and take this general form:

$$\dot{q} = \sum_{i=1}^{m} f_i(q) u_i \tag{9}$$

where $q \in \mathbb{R}^n$ and $u = (u_1, u_2, u_3, \ldots, u_m)^T \in \mathbb{R}^m$ denote the state and the control input of the system, respectively. Let a matrix P be formed by all the vector fields f_is. A compact form of system (9) is as follows:

$$\dot{q} = P(q)u \tag{10}$$

In the literature, the stabilisation problem of (10) has been studied extensively, including the results of Pomet (1992). Consequently, if the vectors $f_1(0), f_2(0), f_3(0), \ldots, f_m(0)$ are linearly independent, then (10) failed Brokett's necessary conditions (Brokett, 1983). Hence, the system cannot be stabilised by a stationary feedback law depending only on the system states. As an alternative, a time-varying control law may guarantee the stability of the system at the origin (see also Beji et al. (2003) for drift systems). The unicycle's example given earlier shows the time-varying stabilisation case. Now, adding the regulation control input, our main result is given by the following theorem.

Theorem 3.1: *Let $D \subset \mathbb{R}^n$ be a set that contains the equilibrium. One considers q a solution of system (10) and $V : \mathbb{R}^n \times [0, +\infty[\to \mathbb{R}$ the Lyapunov function associated to $u_a(q,t) \in \mathbb{R}^m$, satisfying the following:*

$$\begin{aligned} &\alpha_1(q) \le V(q,t) \le \alpha_2(q) \\ &\frac{\partial V}{\partial t} + \frac{\partial V}{\partial q} P(q) u_a(q,t) \le -\alpha_3(q) \end{aligned} \tag{11}$$

Such that for $(q,t) \in D \times [0,+\infty[$*,* α_1*,* α_2 *and* α_3 *are continuous and positive definite functions in D. For all given function* $\nu : \mathbb{R}^n \to \mathbb{R}$ *continuous in D, the control law*

$$u = u_a(q,t) + \nu\left[\left[\left(\frac{\partial V}{\partial q}\right)^t P(q)\right]^T\right]^\perp \tag{12}$$

led to the uniform asymptotic stability towards a given target point of (10).

Proof: As the Lyapunov function V verifies the conditions (11), hence, the control input u_a for $\dot{q} = P(q)u_a(q,t)$ implies its uniform asymptotic stability. Using the same function for system (10) with the control (12), under the hypothesis that the inverse of $P(q)P^T(q)$ exists for $q \in \mathbb{R}^n$, we get:

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial q}\right)^T P(q)u \\
&= \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial q}\right)^T P(q)\left[u_a + \nu\left[\left[\left(\frac{\partial V}{\partial q}\right)^T P(q)\right]^T\right]^\perp\right] \\
&= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial q}P(q)u_a
\end{aligned} \tag{13}$$

which leads to the inequalities in (11). Consequently, for q solution of (10) under the control input (12), the proposed function V verifies:

$$\begin{gathered}
\alpha_1(q) \le V(q,t) \le \alpha_2(q) \\
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial q}P(q)(u_a(q,t) + \nu\left[\left[\left(\frac{\partial V}{\partial q}\right)^T P(q)\right]^T\right]^\perp) \le -\alpha_3(q)
\end{gathered} \tag{14}$$

As a result, (10) and (12) lead to a uniform asymptotic stability result. This ends the proof.

For the existence of a stationary feedback law associated to (10), we propose a similar result given by the following theorem.

Theorem 3.2: *Let* $D \subset \mathbb{R}^n$ *be a set that contains the equilibrium. Let* $u_a(q)$ *an existent stationary feedback law for the equilibrium of (10) and* $V(q)$ *the Lyapunov function associated to* $\dot{q} = P(q)u_a$*. For all scalar function* ν*:* $\mathbb{R}^n \to \mathbb{R}$*, continuous in D, the control input*

$$u = u_a + \nu\left[\left[\left(\frac{\partial V}{\partial q}\right)^T P(q)\right]^T\right]^\perp \tag{15}$$

ensures the asymptotic convergence of solutions of (10) towards the equilibrium.

Proof: Following Theorem 3.1, (10) is asymptotically stable under u_a. Consequently, from Kurzweil (1963), there exists a Lyapunov function V associated to $\dot{q} = P(q)u_a$, meaning that $\dot{V} = \left(\frac{\partial V}{\partial q}\right)P(q)u_a < 0$. Taking the same function for system (10) and the control input (15), then

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial q}P(q)u \\ &= \frac{\partial V}{\partial q}P(q)\left[u_a + \nu\left[\left[\left(\frac{\partial V}{\partial q}\right)^T P(q)\right]^T\right]^\perp\right] \\ &= \frac{\partial V}{\partial q}P(q)u_a < 0\end{aligned} \tag{16}$$

where we have considered that the inverse of $P(q)P^T(q)$ exists. The solutions of (10) converge asymptotically to the desired ones.

4 Regulation control input for systems with drift

Systems with a drift term are also affine control systems to inputs:

$$\dot{q} = \sum_{i=1}^{m} f_i(q)u_i + g(q) \tag{17}$$

where $q \in \mathbb{R}^n$ and $u = (u_1, u_2, u_3, \ldots, u_m)^T \in \mathbb{R}^m$ represent the vector of states and inputs, respectively. Analogy to the case of control systems without drift can be made with a matrix P formed by the vector fields f_is. Hence, the compact form of (17) is as:

$$\dot{q} = P(q)u + g(q) \tag{18}$$

The following results can be easily extended to systems with time invariant stabilising control inputs. Let us introduce the stationary case.

Theorem 4.1: *Let q be a solution of (18), $V : \mathbb{R}^n \times [0, +\infty[\to \mathbb{R}$ the Lyapunov function associated to $u_a(q,t) \in \mathbb{R}^m$ must satisfy the following inequalities:*

$$\begin{aligned}\alpha_1(q) \leq V(q,t) \leq \alpha_2(q) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial q}(P(q)u_a(q,t) + g(q)) \leq -\alpha_3(q)\end{aligned} \tag{19}$$

with $\forall (q,t) \in D \times [0, +\infty[$, α_1, α_2 and α_3 are continuous and positive definite functions in D. Any function $\nu : \mathbb{R}^n \to \mathbb{R}$ added to

$$u = u_a(q,t) + \nu\left[\left[\left(\frac{\partial V}{\partial q}\right)^t P(q)\right]^T\right]^\perp \tag{20}$$

ensure the uniform asymptotic stability of (18) for a predefined target point.

Proof: Substitute (10) by (18) and the second condition of (11) by (19), the procedure is similar to details of Theorem 3.1. If there exists a stationary control input u_a for system (18) with the associated Lyapunov function $V(q)$ then the results remain identical to Theorem 3.2. It means that the control input (15) will guarantee the asymptotic stability of (18).

5 Control-regulation conditions avoiding a set of points

Recall that in the literature, the results were concentrated on the development of the control input u_as which enures the stability of the system around a fixed position or trajectory. In our case, we assume that u_a exists, hence, the equilibrium stability of the undisturbed system is asserted. However, to ensure that the system's solutions avoid some undesirable set $\mathcal{O}$, some conditions on the *regulating control input* ν will be defined taking the system initial conditions in $\mathbb{R}^n \backslash \mathcal{O}$.

To do, let us recall the following. The general form of the controlled system is as follows:

$$\dot{q} = \mathcal{X}(q, \nu) \tag{21}$$

where $q \in \mathbb{R}^n$ and ν is the regulation control input scalar function. The driftless system case leads to (Theorems 3.1 and 3.2):

$$\begin{aligned} \dot{q} &= P(q)\left[u_a(q) + \nu\left[\left[\left(\frac{\partial V}{\partial q}\right)^t P(q)\right]^T\right]^\perp\right] \\ &\triangleq \mathcal{X}(q, \nu) \end{aligned} \tag{22}$$

The time-varying case $\mathcal{X}$ evolves as function of (q, ν, t). The system with drift is considered as follows (Theorem 3.1):

$$\begin{aligned} \dot{q} &= P(q)[u_a(q) + \nu\left[\left[\left(\frac{\partial V}{\partial q}\right)^t P(q)\right]^T\right]^\perp\right] + g(q) \\ &\triangleq \mathcal{X}(q, \nu) \end{aligned} \tag{23}$$

Proposition 5.1: *Let us consider the system (21), which evolves in $\mathbb{R}^n$. For a continuous $\varphi : E \in \mathbb{R}^n \to F \in \mathbb{R}$ and A as a compact set, one defines the set of points to be avoided:*

$$\mathcal{O} = \varphi^{-1}(A). \tag{24}$$

Let N a submanifold in $\mathbb{R}^n \backslash \mathcal{O}$, surrounding $\mathcal{O}$ (i.e., if U is in neighborhood of a point of $\partial\mathcal{O}$, then $N \cap U \neq \emptyset$). If there exists a function $\nu(q)$ such that

$$\varphi(b + \tau\mathcal{X}(b, \nu)) \in C_F\mathring{A} \tag{25}$$

for all $\tau \in [0, 1]$ and $b \in N \cup \partial\mathcal{O}$, we have the following,

1 *The integral curve of $\mathcal{X}(q, \nu)$ from $q_0 = q(t_0) \in N \cup \partial\mathcal{O}$ avoids $\mathcal{O}^\circ$.*

2 *If further $\mathcal{X}$ is locally Lipchitzian only on $N \cup \partial\mathcal{O}$, then the integral curve of $\mathcal{X}(q,\nu)$ from $q_0 = q(t_0) \in \mathbb{R}^n \backslash \mathcal{O}$ do not leave $\mathbb{R}^n \backslash \mathcal{O}$.*

Proof: If we want to study the value of the real solution at all $T > 0$, we assume that we are working on an interval $[0, T]$ we divided into subintervals $[t_k, t_{k+1}]$ of length $h = \frac{T}{n}$ for any integer $n > 0$, with $t_k = kh$ for $k = 0, \ldots, n$ and $t_0 = 0$. Consider a finite sequence $(y_k)_{k=0,\ldots,n}$ defined by recurrence relation

$$y_{k+1} = y_k + \lambda h \mathcal{X}(y_k, \nu_k) \tag{26}$$

for $k = 0, \ldots, n-1$ and $y_0 = q_0 \in N \cup \partial\mathcal{O}$, and we consider $q_n : [0, T] \to \mathbb{R}^2$ the piecewise affine function defined by $q_n(t_k) = y_k$ for all $k \in \mathbb{N}$ (i.e., $q_n(t_k + \lambda h) = y_k + \lambda h \mathcal{X}(y_k, \nu_k)$, $\lambda \in [0, 1]$).

Let us prove by recurrence that the sequence (y_k) avoids $\mathcal{O}^\circ$ for $k = 0, \ldots, n-1$: Let $y_0 \in N \cup \partial\mathcal{O}$ then according to the hypothesis of the proposal, there exists a ν which verifies $\varphi(y_0 + \lambda h \mathcal{X}(y_0, \nu_0)) \in C_F A^\circ$ where $\nu(y_k) = \nu_k$ for $k = 0, \ldots, n-1$ hence $\varphi(y_1) \in C_F A^\circ \Rightarrow y_1 \in C\mathcal{O}^\circ$ then

$$\begin{cases} y_1 \in N \cup \partial\mathcal{O} \\ or \\ y_1 \in C(N \cup \mathcal{O}). \end{cases} \tag{27}$$

If you are in the first case where $y_1 \in N \cup \partial\mathcal{O}$ then according to the previous proposal $y_2 \in C\mathcal{O}^\circ$. If you're in the latter case then $y_2 = y_1 + \lambda h \mathcal{X}(y_1, \nu_1)$ where $\lambda \ll 1$ remains in $C(N \cup \mathcal{O})$ or enters in $N \cup \partial\mathcal{O}$, which implies that y_2 is also in $C\mathcal{O}^\circ$. This is because if we assume by contradiction that $y_2 \in \mathcal{O}^\circ$ then there exists an $r > 0$ such that $B(y_2, r) \subset \mathcal{O}^\circ$. But if we consider a $y \in B(y_2, r)$ we have:

$$\|y - y_2\| = \|y - y_1 - \lambda h \mathcal{X}(y_1, \nu_1)\| < r. \tag{28}$$

Passing to the limit when $\lambda \to 0$, the continuity of the norm, implies that $\|y - y_1\| \leq r$ which leads to $B(y_2, r) \subset \bar{B}(y_1, r)$ hence $B(y_2, r) \subset B(y_1, r)$. But as these two balls have the same radius, this implies that $y_1 = y_2 \in \mathcal{O}^\circ$ which contradicts the hypothesis of the second case.

Similarly if $y_k \in C\mathcal{O}^\circ$, then $y_{k+1} \in C\mathcal{O}^\circ$.

Hence the sequence $(y_k) \subset C\mathcal{O}^\circ$ and $q_n(t) \in C\mathcal{O}^\circ$ for all $t \in [0, T]$. This is true for all n and it is well known (according to the Euler scheme) that the sequence $(q_n(.))$ converges uniformly on $[0, T]$ towards the solution of system (21) from the $q_0 \in N \cup \partial\mathcal{O}$.

As $C\mathcal{O}^\circ$ is closed it means that the limit $q(t) \in C\mathcal{O}^\circ$ for all $t \in [0, T]$. This is true for all $T > 0$, hence the integral curve of system (21) avoids $\mathcal{O}^\circ$.

Now, if the vector field $\mathcal{X}(q, \nu)$ is locally Lipchitzian only on $N \cup \partial\mathcal{O}$, the theorem of Cauchy-Lipschitz guarantees that the solution is unique in $N \cup \partial\mathcal{O}$. Hence, if an integral curve $\gamma(t)$ of $\mathcal{X}(q, \nu)$, starting from $q(t_0) \in \mathbb{R}^n \backslash \mathcal{O}$, goes in $N \cup \partial\mathcal{O}$, then $\gamma(t)$ restricted to $N \cup \partial\mathcal{O}$, coincides with one of curves starting from $N \cup \partial\mathcal{O}$. Then $\gamma(t)$ avoids $\mathcal{O}^\circ$.

6 Multi-mobile robot control and regulation

One considers a multi-mobile robot system in formation under a shared information. Unlike the case of a single robot, the convergence of the formation must be to a set of points and this complicates the control task. Our stability results are based on the invariant principle of LaSalle, and the system's convergence involves multiple objectives. This last is built around the target and must avoid the set $\mathcal{O}$.

Let us recall the kinematics of n vehicles:

$$\dot{q} = u \tag{29}$$

where $q \in \mathbb{R}^{2n}$ regroups the states and $u \in \mathbb{R}^{2n}$ is the set of inputs. The proposed approach will subsequently be extended to multi-agent systems cases while for a single robot, it implies the stabilisation towards only one target. Similarly, we will take into account the formation's navigation in a disturbed environment by obstacles and consider collisions between agents. Consequently, the stabilising controller will be increased by other terms that ensure the regulation of each vehicle trajectory and maintain the formation's stability.

Definition 6.1 (Invariant Set): *Ω is an invariant set for the system $\dot{x} = f(x)$ if each trajectory $x(t)$ starting in Ω remains over time in Ω.*

Theorem 6.2: *For the system (29) where $q \in \Omega \subset \mathbb{R}^{2n}$, assume that $\exists u_a / u = u_a$ stabilises (29) in a free environment and*

$$\Omega = \{q \in \mathbb{R}^{2n} / 0 \leq V(q) \leq p\} \tag{30}$$

is an invariant set with respect to $\dot{q} = u_a$, and $V : \mathbb{R}^{2n} \to \mathbb{R}_+$ such that $\nabla_q V u_a \leq 0$. The control input for the multi-vehicle formation,

$$u = u_a - \begin{pmatrix} \nu_1(q) & 0 & \dots \\ 0 & \ddots & \\ \vdots & 0 & \nu_n(q) \end{pmatrix} \otimes I_2 \begin{pmatrix} \begin{pmatrix} (\nabla_q V)_{x_1} \\ (\nabla_q V)_{y_1} \end{pmatrix}^{\perp} \\ \vdots \\ \begin{pmatrix} (\nabla_q V)_{x_n} \\ (\nabla_q V)_{y_n} \end{pmatrix}^{\perp} \end{pmatrix} \tag{31}$$

ensures that the states of (29) converge to the set Ω. $\otimes$ denotes the Kronecker product, I_2 the identity matrix $\in \mathcal{M}_{2\times 2}(\mathbb{R})$, $\nu_i : \mathbb{R}^{2n} \to \mathbb{R}$, and $\nabla_q V = [(\nabla_q V)_{x_1}, (\nabla_q V_{y_1}, \dots, (\nabla_q V)_{x_n}, (\nabla_q V)_{y_n}]$.

Proof: The set $\Omega = \{q \in \mathbb{R}^{2n} / 0 \leq V(q) \leq p)\}$ is invariant with respect to $\dot{q} = u_a$ and V verifies $\dot{V} = \nabla_q V u_a \leq 0$. The LaSalle's theorem (see Khalil, 2001) implies that the solutions converge to the great invariant set $E = \{q \in \mathbb{R}^{2n}/\dot{V} = 0\}$. Using the same function V for (29,31), the time derivative is given by:

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial q}\dot{q} \\ &= \frac{\partial V}{\partial q}u_a - \frac{\partial V}{\partial q}\mathfrak{M} \otimes I_2 \begin{pmatrix} F_1^{\perp} \\ \vdots \\ F_n^{\perp} \end{pmatrix}\end{aligned} \tag{32}$$

with $F_i = \begin{pmatrix} (\nabla_q V)_{x_i} \\ (\nabla_q V)_{y_i} \end{pmatrix}$. $\mathfrak{M}$ is a diagonal matrix whose terms are the components of $\nu = [\nu_1(q), \nu_2(q), \ldots \nu_n(q)]$. The matrix corresponding to $\mathfrak{M} \otimes I_2$ is given by:

$$\mathfrak{M} \otimes I_2 = \begin{pmatrix} \mathfrak{A}_1 & 0 & \ldots \\ 0 & \ddots & \vdots \\ \vdots & 0 & \mathfrak{A}_n \end{pmatrix}; \quad \mathfrak{A}_i = \begin{pmatrix} \nu_i & 0 \\ 0 & \nu_i \end{pmatrix}$$

The quantity

$$\frac{\partial V}{\partial q}\mathfrak{M} \otimes I_2 \begin{pmatrix} F_1^{\perp} \\ \vdots \\ F_n^{\perp} \end{pmatrix} = \sum_{i=1}^{n} F_i^t \mathfrak{A}_i F_i^{\perp} = \nu_i F_i^t F_i^{\perp} = 0 \tag{33}$$

It is obvious to show,

$$\dot{V} = \frac{\partial V}{\partial q}u_a \leq 0 \tag{34}$$

Hence, $\Omega = \{q \in \mathbb{R}^{2n}/0 \leq V(q) \leq V(q_0)\}$ is invariant for the solutions of (29), (31). Thus, the system (29), (31) states converge to the great invariant set of E. This ends the proof.

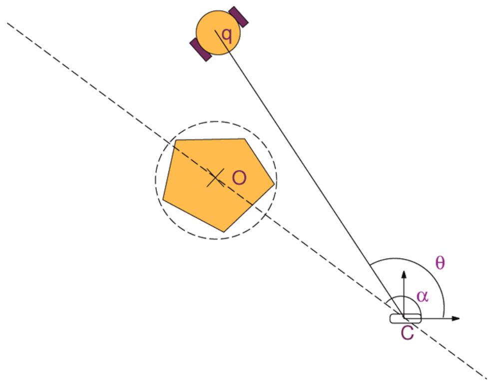
7 Application: decentralised control for the rendezvous problem

Let us consider a family of n agents in $2D$-space, and a target $C = (C_x, C_y)$, assumed to be fixed. To C we attach a fixed frame, which is considered as an inertial frame. Let $O = (O_x, O_y)$ denotes the coordinate of an unmoving obstacle (Figure 1). The agents' kinematics is described by:

$$\dot{q} = u \tag{35}$$

where $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^{2n}$ and $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^{2n}$. From Theorem 6.2, our aim is to find a control u_a that stabilises the formation and express the regulation control input ν, which permits to avoid some localised obstacles.

Figure 1 Agent parameterised by angles θ and α (see online version for colours)



7.1 *Stabilising control input*

For system (35), the stabilising control input vector is formed by each agent's stabilising input vector,

$$u_a = (u_{a1}, u_{a2}, \ldots, u_{an})$$

Proposition 7.1: *For n agents with kinematics (35), the stabilising controller*

$$u_i = u_{ai} \tag{36}$$

where

$$u_{ai} = -(\| q_i - C \|^2 - l^2)(q_i - C) \tag{37}$$

and the initial position $q_{i0} \in \Omega$ with $\Omega \subset \mathbb{R}^{2n}$, and

$$\Omega = \{q \in \mathbb{R}^{2n} / l \leq \| q_i - C \| \leq K\}$$

$u_a = (u_{a1}, u_{a2}, \ldots, u_{an})$ *ensures that the solutions of (35), (37) converge towards the set M:*

$$M = \{q \in \Omega / \| q_i - C \| = l\}$$

with $K \geq \| q_{i0} - C \|$.

The proof of Proposition 7.1 is based on LaSalle's theorem. The following lemmas are introduced such that Lemma 7.2 permits to verify that Ω is invariant to (35), (37). Lemma 7.3 shows the adequate decreasing function V. Finally, Lemma 7.4 determines the great invariant set M, which is also the equilibrium set of the system.

Lemma 7.2: *The $\Omega = \{q \in \mathbb{R}^{2n} / l \leq \| q_i - C \| \leq K\}$ set is invariant over time for (35), (37). Furthermore, Ω is compact.*

Proof: Assume that $q_0 \in \Omega$ and consider the function

$$S(q_i) = (\| q_i - C \|^2 - l^2) \tag{38}$$

The time derivative of $S(q_i)$ throughout the trajectory of (35), (37) is given by

$$\begin{aligned} \dot{S}(q_i) &= 2 < \dot{q}_i, q_i - C > \\ &= 2 < u_i, q_i - C > \\ &= -2(\| q_i - C \|^2 - l^2) \| q_i - C \|^2 \\ &= -2S(q_i)(S(q_i) + l^2) \end{aligned} \tag{39}$$

Consequently,

$$\frac{S(q_i)}{S(q_i) + l^2} = \frac{S(q_{i0})}{S(q_{i0}) + l^2} exp(-2l^2(t - t_0)) \tag{40}$$

As $S(q_{i0}) = \| q_{i0} - C \|^2 - l^2 \geq 0$ This is from $q_{i0} \in \Omega$, then

$$S(q_i) = \| q_i - C \|^2 - l^2 \geq 0 \Leftrightarrow \| q_i - C \| \geq l \tag{41}$$

On the other hand, let $F(q_i)$ such that:

$$F(q_i) = \| q_i - C \|^2 \tag{42}$$

The differential of F with respect to time t is as:

$$\begin{aligned} \dot{F}(q_i) &= 2 < \dot{q}_i, q_i - C > \\ &= 2 < u_i, q_i - C > \\ &= -2(\| q_i - C \|^2 - l^2) \| q_i - C \|^2 \end{aligned} \tag{43}$$

It is obvious that $\dot{F}(q_i) \leq 0$, then F is a decreasing function. This means that

$$\| q_i - C \| \leq \| q_{i0} - C \| \leq K \tag{44}$$

As a result, if $q_0 \in \Omega$ then $q \in \Omega$.

Lemma 7.3: *Consider the vector q with formed by solutions of (35), (37). The differential of the following function V with respect to t*

$$V(q) = \sum_{i=1}^{n} (\| q_i - C \|^2 - l^2) \tag{45}$$

is negative with respect to the set Ω,

Proof: The time derivative of V throughout the trajectory of (35) with the given control input in Proposition 7.1 leads to:

$$\begin{aligned} \dot{V}(q) &= \sum_{i=1}^{n} < \dot{q}_i, q_i - C > \\ &= -\sum_{i=1}^{n} (\| q_i - C \|^2 - l^2) \| q_i - C \|^2 \end{aligned} \tag{46}$$

As $q_0 \in \Omega$, and Ω is invariant for (35), (37), then $q \in \Omega$. As a result $(\| q_i - C \|^2 - l^2) \geq 0$, consequently $\dot{V}(q) \leq 0$.

Lemma 7.4: *Consider the vector q with its components the solutions of (35), (37), then*

$$M = \{q \in \Omega / \| q_i - C \| = l\} \tag{47}$$

is the great invariant set in $E = \{q \in \Omega / \dot{V} = 0\}$.

Proof: From the time derivative of V reduced to zero,

$$\dot{V}(q) = -\sum_{i=1}^{n} (\| q_i - C \|^2 - l^2) \| q_i - C \|^2 = 0$$

we get

$$E = \{\| q_i - C \| = l\} = M \tag{48}$$

As $q \in \Omega$, let $q_0 \in M$ and $S(q_i) = \| q_i - C \|^2 - l^2$, from the proof of Lemma 7.2,

$$\frac{S(q_i)}{S(q_i) + l^2} = \frac{S(q_{i0})}{S(q_{i0}) + l^2} exp(-2l^2(t - t_0)) \tag{49}$$

$q_0 \in M$ then $S(q_{i0}) = 0$, which implies that $S(q_i) = \| q_i - C \|^2 - l^2 = 0$. Consequently, $q \in M$ representing the great invariant set of E. The above-mentioned results contribute to the proof of Proposition 7.1 and it will be achieved in the following step.

Proof (Proposition 7.1): Following proofs given in Lemmas 7.2, 7.3, and 7.4, Ω is invariant with respect to (35), (37). Furthermore, $\dot{V} \leq 0$ in Ω. From LaSalle's theorem, each solution of (35), (37) that admits initial conditions in Ω converge to M as $t \to \infty$. The set M is given by Lemma 7.4, which is the great invariant set of $E = \{q \in \Omega / \dot{V} = 0\}$. This ends the proof.

Our first constat is achieved by the expressions given to the control input $u_a = [u_{a_1}, u_{a_2}, \ldots, u_{a_n}]$ (Proposition 7.1), i.e., the convergence of each agent to a circle surrounding the target without any motion planning. It remains to ensure that the formation does not reenter in collision with obstacles.

7.2 Regulation control input for the formation

In the following theorem, we give a regulation control input that ensures the convergence of each agent to a point on a circle, considered as an attractive set, centred by the target. The avoidance of obstacles in the plan will be also considered.

Theorem 7.5: *Consider the set*

$$\Omega = \{q \in \mathbb{R}^{2n} / l \leq \| q_i - C \| \leq K\}$$

The n agents are represented by system (35), defined in Ω. Let $q_0 = (x_0, y_0)$ denotes the initial positions at time $t = t_0$, and $L(x) = \frac{O_y}{O_x}x$ be the function associated to the line joining the centre of the attractive set $C(C_x, C_y)$ and $O(O_x, O_y)$. O denotes the centre that surrounds the obstacle. The control input,

$$u_i = u_{ai} + \nu_i(q_i - C)^{\perp} \tag{50}$$

where

$$u_{ai} = -(\| q_i - C \|^2 - l^2)(q_i - C) \tag{51}$$

and

$$\nu_i = -\frac{sign([y_{i0} - L(x_{i0})][C_x - O_x])}{\|q_i - O_{q_i}\|}(\| q_i - C \|^2 - l^2) \tag{52}$$

for all initial conditions in Ω, converge the solutions of (35), (50), (51), (52) towards M with

$$M = \{q \in \Omega / \| q_i - C \| = l\}$$

and $K \geq \| q_{i0} - C \|$. Moreover, the ith agent avoids the time-varying point O_{q_i}.

Proof: We have

$$\begin{aligned} \Omega &= \{q \in \mathbb{R}^{2n}/l \leq \| q_i - C \| \leq K\} \\ &= \{q \in \mathbb{R}^{2n}/0 \leq \| q_i - C \|^2 - l^2 \leq K^2 - l^2\} \\ &= \{q \in \mathbb{R}^{2n}/0 \leq V(q) \leq p \triangleq K^2 - l^2\} \end{aligned} \tag{53}$$

From Lemmas 7.2 and 7.3, Ω is invariant, $\dot{V} \leq 0$ with respect to (35), (37), and $\frac{\partial V}{\partial q} = q - C$. Following Theorem 6.2 the system's (35), (50), (51), (52), (35) and (37) solutions approach the same sets. On the other hand, following Proposition 7.1, the solutions of (35,37) converge towards M, consequently the system's solutions (35), (50), (51), (52) converge towards M. M is none other than the circle of centre C and radius r.

It remains to prove that the ith agent avoids O_{q_i}. To do, we have to analyse the results of Proposition 5.1. In closed loop, the decentralised control including the regulation problem is expressed by:

$$\begin{aligned} \dot{q} &= -(\| q - C \|^2 - l^2)(q - C) + \nu(q - C)^{\perp} \\ &\triangleq \mathcal{X}(q, \nu(q)) \end{aligned} \tag{54}$$

where ν is given by (52). It is obvious to show that for all $q \in N = \{q \in \mathbb{R}^2/r < \|q - O\| < \varepsilon\}$ with $\varepsilon < \|O\|$ and for all $\tau \in [0, 1]$, q verifies the following inequality:

$$\|q - O + \tau\mathcal{X}(q, \nu)\| \geq \|q - O\| > r$$

Consequently, $\forall\, P \in \mathcal{O}$, $\mathcal{O}$ denotes the disc of centre O and radius r,

$$P \neq q + \tau\mathcal{X}(q,\nu) \Leftrightarrow q - P \neq -\tau\mathcal{X}(q,\nu)$$

From Proposition 5.1, the function ν ensures that the integral curve $\mathcal{X}(q,\nu)$ resulting from $q(t_0) \in N$ is in N for a time sufficiently small, and returns in the variety $\mathbb{R}^2\backslash\mathcal{O}$. Furthermore, it is straightforward to prove that $\mathcal{X}(q,\nu)$ is Lipschitzian in N, consequently Proposition 5.1 guarantees that all integral curves resulting from $q(t_0) \in \mathbb{R}^2\backslash\mathcal{O}$ remain in $\mathbb{R}^2\backslash\mathcal{O}$. This ends the proof.

In addition to the stability and regulation results presented earlier, the following corollaries guarantee that each agent's trajectory obeys to the invariance property of each navigation sub-plan. We have to prove the following: if an agent is initialised into a region defined by the half-plan, over time, it remains there until the target is reached. Furthermore, the obstacle avoidance will be ensured within the adequate region.

Corollary 7.1: *The following two subsets K and H, dividing the navigation plan into two navigation's regions, are invariant with respect to systems (35), (50), (51), (52),*

$$\begin{aligned} K_I &= \{q(x,y) \in \Omega / y \geq L(x)\} \\ H_I &= \{q(x,y) \in \Omega / y < L(x)\} \end{aligned} \tag{55}$$

$L(x)$ is the line that joints the target and the obstacle and $\Omega = \{q \in \mathbb{R}^{2n}/0 \leq V(q) \leq p\}$.

Proof: From (50)–(52), the i^{th} agent's kinematic (35) becomes:

$$\dot{q}_i = -q_i(\| q_i \|^2 - l^2) + \nu_i q_i^{\perp} \tag{56}$$

Using the polar coordinate transformation, system (56) obeys to the following differential system:

$$\begin{aligned} \dot{\rho}_i &= -(\rho_i^2 - l^2)\rho_i \\ \dot{\theta}_i &= \nu_i \end{aligned} \tag{57}$$

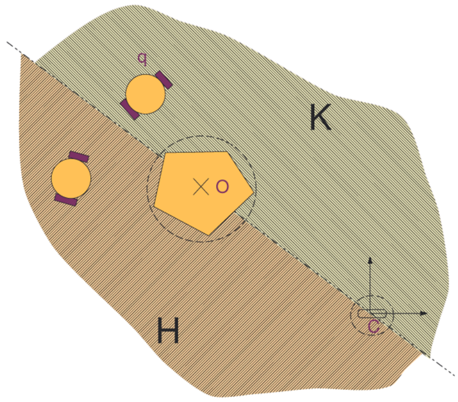
The θ_i state represents the angle from $\overrightarrow{Cq_i}$ and the horizontal (Figure 2). Also, one introduces α, which denotes the angle between $L(x)$ and the horizontal. From (57), the behaviour of θ depends on terms in $sign(.)$ function of (52). One distinguishes 4 cases.

-1st case If $C_x \geq O_x$, one obtains 2 cases

1/ if $y_{i0} \geq L(x_{i0})$ $\dot{\theta}_i = -\frac{\rho_i^2 - l^2}{|\sqrt{(\rho_i\cos\theta_i - O_x)^2 + (\rho_i\sin\theta_i - O_y)^2} - r|} \leq 0$, then $\theta_i \leq \theta_{i0}$. On the other hand, as $y_{i0} \geq L(x_{i0})$, which implies that $\theta_i \leq \theta_{i0} \leq \alpha$, then $\forall\, t$, the state $q(t)$ remains into the half superior plan defined by $L(x)$. One proves that $y_i \geq L(x_i)$.

2/If $y_{i0} \leq L(x_{i0})$ then $\dot{\theta}_i \geq 0$, consequently, $\theta_i \geq \theta_{i0}$. Furthermore, $y_{i0} \leq L(x_{i0})$, which implies that $\theta_i \geq \theta_{i0} \geq \alpha$, i.e., q belongs to the half superior plan defined by $L(x)$. One proves $y \leq L(x)$.

Figure 2 The invariant subsets K and H (see online version for colours)



-2nd case If $C_x \leq O_x$

Similarly, we emphasise two cases.

3/ If $y_{i0} \geq L(x_{i0})$, $\dot{\theta}_i \leq 0$, and the result given by $1/$ in the **1stcase**, we have $y_i \geq L(x_i)$.

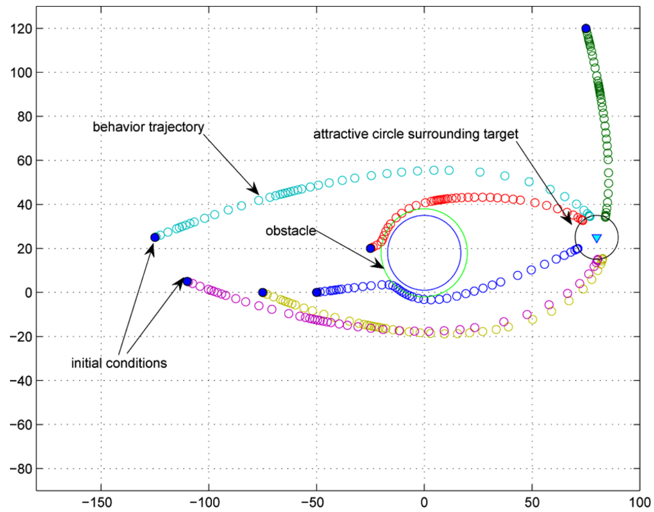
4/ If $y_{i0} \leq L(x_{i0})$, the same analysis can be adopted as given in **2/-1stcase**, then $y_i \leq L(x_i)$.

These four cases lead to the following, if $y_{i0} \geq L(x_{i0})$ (resp. $y_{i0} < L(x_{i0})$) then $y_i \geq L(x_i)$ (resp. $y_i < L(x_i)$) $\forall t \in [0, +\infty[$. Finally, one concludes that K and H are invariant for the system (35), (50), (51), (52). The proposed regulation control input ensures that the agents avoid the obstacles while staying on the invariant subset defined earlier. These subsets are determined with respect to the initial position of each agent.

7.3 Simulation results

To confirm the obtained theoretical results, one simulates the trajectories obtained from (35) under the proposed control law including regulation, given by (50), (51), (52). The analysis of the formation's stability integrates six agents initially scattered in the navigation plan, and one known obstacle is also incorporated in the plan. The simulation is sketched in Figure 3. The six agents avoid the obstacle, while converging to a position near to the target, which is a circle like an attractive set. This shows the effectiveness of results given by Theorem 7.5. The proposed regulation control input ensures that each agent avoids the obstacle while staying on the invariant subset defined by (55). Recall that these subsets are determined with respect to the line connecting the centres of circles containing the obstacle and the target. To preserve collisions between agents, it remains to solve the communication problem. The connection inspired from the graph theory can take into account the behaviour of the multi-agent formation, which is also another problem of regulation and it can take part in the conception of the ν function.

Figure 3 Six agents stabilisation around a triangle like an objective (see online version for colours)



8 Conclusion

For multi-vehicle navigation including regulations with respect to some predefined obstacles, the stabilising control problem is solved analytically in a decentralised form and proved throughout LaSalle's invariance principle and the Lyapunov theory. Compared with a single robot's stabilisation problem, which implies the convergence towards only one target, the group's stabilisation is reasserted towards a set of targets. Under the proposed regulation control-input, the set's invariance is shown and it can be adjusted with respect to the objective and the navigation environment. After initially scattered in the navigation plan, each agent moves towards the attractive set circumscribing the target while it avoids the obstacle and two subsets subdividing the environment plan are shown invariants. As an agent will move towards the appropriate navigation's subsets, this will limit energy consumptions. It remains to integrate in a novel form of the regulation control-input strong interconnections between agents and collisions.

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