Local finite-time stability and stabilization analysis of interconnected systems

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Abstract: Interconnection is omnipresent in a system through the state variables and induced for multi-system interaction and shared tasks. Typically, the example of multi-agent coordination was studied as an interconnected system. The paper deals with the finite-time stability problem of a general form of interconnection presented as a perturbation term. Sufficient conditions for finite-time stability are derived. A second interest is given to the interaction of multiple controlled autonomous systems, and where the multi-system control-input is established both for finite-time stabilization. As an example of application, the finite-time tracking problem of four unicycles is studied.

Keywords: Finite-time stability, interconnected nonlinear systems, Lyapunov function, homogeneity.

1. INTRODUCTION

Over the past decades, many researchers have been focused on asymptotic or exponential stability of interconnected systems. Also, many results are achieved in the stability analysis of interconnected systems and have been limited to some classes of dynamical systems. For example, the string stability in Swaroop (1996) was described with linear interconnection. Further, necessary and sufficient conditions for stability of linear interconnected systems based on graph Laplacian matrix is presented in Fax (2002). Due to complexity of system models, stability concepts have been extended to more general form of interconnected nonlinear systems Fax (2003) and Khalil (2001).

These results had solved only the asymptotic or exponential stability. For dynamical systems theory, the asymptotic/exponential stability notion involves the convergence of the system trajectories to an equilibrium state which does mean that states convergence in finite settling time. However, the finite-time stability of dynamical systems implies that trajectories converge to an equilibrium state in finite-time. With respect to the classical control theory, the finite-time stability theory is a more practical concept. Haimo (1986) studied autonomous scalar systems and gives necessary and sufficient conditions for finite-time stability of the system's origin. Further, the stability problem in finite-time of nonautonomous systems was treated by several authors such as Orlov (2003) for switched systems and Moulay (2003) proposed sufficient conditions using Lyapunov function. Haddad (2008) provides Lyapunov and converse Lyapunov conditions for finite-time stability. A principal result of finite-time stability for homogeneous nonautonomous systems was obtained in Bhat (1997). In Zoghlami (2012), sufficient conditions are proposed for finite-time stability of homogeneous and T-periodic systems and where the averaging method has led to a perturbed average system.

Indeed, when we investigate the stability of a nonlinear dynamical system, the complexity of the analysis grows quickly as the order of the system increases. This situation pushes us to seek for methods in order to simplify the analysis (see Khalil (2001)).

However, the stability analysis of interconnected systems and the study of controlled systems associated to graph theory, may lead to investigation of multi-agent system. In fact, analysis of multi-agent group has witnessed a large and growing literature concerned with the coordination of multi-mobile autonomous agents including flocking and formation Saber (2007) Saber (2004) Xiaoli (2008) Feng (2009). In this area, the i^{th} agent model is considered as a driftless subsystem, and taken kinematically as a first order ($\dot{x}_i = u_i$, u_i is the input) or dynamically as a second order ($\ddot{x}_i = u_i$) leading to asymptotic or finite-time stability.

In this paper, we will propose to solve the finite-time stability and the stabilization of interconnected systems where each individual subsystem is nonlinear. Each subsystem is considered as finite-time stable with the associated Lyapunov function. Consequently, each nonlinear control law is constructed by taking each subsystem properties and should ensure the interconnection's stability in finite-time. The paper is organized as follows: the second section is devoted to some preliminary mathematical results of finite-time stability and graph theory. In the third section, we present sufficient conditions for stability of interconnected nonlinear systems. Section 4 deals with the finite-time stability of controlled systems. The finite-time tracking of multi-unicycle is presented in section 5. Section 6 concludes the paper.

2. MATHEMATICAL PRELIMINARIES

In this section, we introduce notations, definitions and present some results needed for development of our main approach.

2.1 Finite-time stability

In this section we present several preliminary results and definitions which are related to the problem of finite-time stability of nonautonomous systems.

Definition 1. Moulay (2003) Let us consider a nonautonomous dynamic system of the form:

$$\dot{x} = f(t, x) \tag{1}$$

where f is continuous functions in $\mathbf{R}_{\geq 0} \times \mathbf{R}^n$.
The origin is weakly finite-time stable for the system (1) if:

(1) the origin is Lyapunov stable for the system (1),
(2) for all $t \in I$, where I is nonempty interval of $\mathbf{R}$, there exists $\delta(t) > 0$, such that if $x \in \mathcal{B}_{\delta(t)}$ then for all $\Phi_t^x \in S(t, x)$:
   i) $\Phi_t^x(\tau)$ is defined for $\tau \geq t$,
   ii) there exists $0 \leq T(\Phi_t^x) < +\infty$ such that $\Phi_t^x(\tau) = 0$ for all $\tau \geq t + T(\Phi_t^x)$.
   Let
   $T_0(\Phi_t^x) = \inf\{T(\Phi_t^x) \geq 0 : \Phi_t^x(\tau) = 0 \quad \forall \tau \geq t + T(\Phi_t^x)\}$
(3) Moreover, if $T_0(t, x) = \sup_{\Phi_t^x \in S(t,x)} T_0(\Phi_t^x) < +\infty$, then the origin is finite-time stable for system (1).

$T_0(t, x)$ is called the settling time with respect to initial conditions of system (1).

Theorem 2. Moulay (2003) Suppose that the origin is an equilibrium point i.e $f(t, 0) = 0$ of the system (1).
If there exists a positive definite function $\mathbf{r}$ such, for $\varepsilon > 0$

$$\int_0^\varepsilon \frac{dz}{\mathbf{r}(z)} < \infty$$

If V is a Lyapunov function continuously differentiable such that

$$\dot{V} \leq -\mathbf{r}(V)$$

then the system (1) is finite-time stable.

The following definitions are useful in the case of a nonautonomous homogeneous system. Further details are in Bhat (2005),M.Kawski (1990) and M.Kawski (1999).

Theorem 3. Haddad (2008) Let $\lambda \in (0, 1)$ and let $\mathcal{N}$ an open neighborhood of the origin. Assume that there exists a classe $\mathcal{K}$ function $\nu : [0, r] \to \mathbf{R}_+$, where $r > 0$, such that $B_r(0) \subseteq \mathcal{N}$ and for $t \in \mathbf{R}_+$ and $x \in B_r(0)$

$$\|f(t, x)\| \leq \nu(\|x\|) \tag{2}$$

If the zero solution $x(t) \equiv 0$ to (1) is uniformly finite-time stable and the settling-time function $T(., .)$ is jointly continuous at $(t, 0)$ for $t \geq 0$, then there exist a class $\mathcal{K}$ function $\alpha(.)$, a positive constant $k > 0$, a continuous function $V : [0, \infty) \times \mathcal{N} \to \mathcal{R}$, and a neighborhood $\mathcal{M}$ of the origin such that $\dot{V}(t, x)$ is defined for $(t, x) \in [0, \infty) \times \mathcal{R}$ and
$V(t, 0) = 0$; $t \in [0, \infty)$,
$\alpha(\|x\|) \leq V(t, x)$, $t \in [0, \infty)$, $x \in \mathcal{M}$,
$\dot{V}(t, x) \leq -k(V(t, x))^\lambda$, $t \in [0, \infty)$, $x \in \mathcal{M}$.

Definition 4.
- The dilation is considered of the form
  $$\Delta_\varepsilon(x_1, \ldots, x_n) = (\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n) \tag{3}$$
  where $x_1, \ldots, x_n$ are suitable coordinates on $\mathbf{R}^n$ and $r_1, \ldots, r_n$ are positive real numbers. The dilation corresponding to $r_1 = \ldots = r_n = 1$ is the standard dilation in $\mathbf{R}^n$.
- The Euler vector field of the dilation is linear and is given by
  $$\nu = r_1 x_1 \partial x_1 + \ldots + r_n x_1 \partial x_n$$
- A function $f : \mathbf{R}^n \to \mathbf{R}$ is homogeneous of degree l with respect to the dilation (3) if and only if
  $$f(\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n) = \varepsilon^l V(x_1, \ldots, x_n)$$
- Consider n-dimensional system
  $$\dot{x} = f(x), \quad x = (x_1, ..., x_n)^T$$
  a vector field $f(x) = (f_1(x), ..., f_n(x))^T$ is homogeneous of degree $m \in \mathbf{R}$ with dilation (3) if
  $$f_i(\varepsilon^{r_1} x_1, ..., \varepsilon^{r_n} x_n) = \varepsilon^{m+r_i} f_i(x) \quad i = 1, ..., n$$
- The system,
  $$\dot{x} = f(x) + \hat{f}(x), \quad \hat{f}(0) = 0 \tag{4}$$
  is called locally homogeneous if f is homogeneous of degree $m \in \mathbf{R}$ with dilation (3) and $\hat{f}$ is continuous vector field satisfying
  $$\lim_{\varepsilon \to 0} \frac{\hat{f}_i(\varepsilon^{r_1} x_1, ..., \varepsilon^{r_n} x_n)}{\varepsilon^{m+r_i}} = 0 \quad \forall x \neq 0;\ i = 1, ..., n$$
- A continuous map from $\mathbf{R}^n$ to $\mathbf{R}$, $x \mapsto \rho(x)$ is called a homogeneous norm with respect to the dilation Δ_ε i.e:
  1) $\rho(x) \geq 0$, $\rho(x) = 0 \Leftrightarrow x = 0$;
  2) $\rho(\Delta_\lambda x) = \lambda \rho(x)$ $\forall \lambda > 0$
- The homogeneous norm may always be defined as
  $$\rho(x) = |x_1^{\frac{c}{r_1}} + x_2^{\frac{c}{r_2}} + ... x_n^{\frac{c}{r_n}}|^{\frac{1}{c}}$$
  where c is some positive integer evenly divisible by r_i

2.2 Graph theory

In this subsection, we introduce some basic concepts in algebraic graph theory for multi-agent networks. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a directed graph, where $\mathcal{V} = \{0, 1, 2, ..., n\}$ is the set of nodes, node i represents the ith agent, $\mathcal{E}$ is the set of edges, and an edge in $\mathcal{G}$ is denoted by an ordered pair (i, j). $(i, j) \in \mathcal{E}$ if and only if the ith agent can send

information to the jth agent directly.
$A = [a_{ij}] \in \mathbf{R}^{n+1\times n+1}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements, where $a_{ij} > 0$ if there is an edge between the ith agent and jth agent and $a_{ij} = 0$ otherwise. In this paper, we will refer to graphs whose weights take values in the set$\{0, 1\}$ as binary and those graphs whose adjacency matrices are symmetric as symmetric. Let $D = diag\{d_0, d_1, ..., d_n\} \in \mathbf{R}^{n+1\times n+1}$ be a diagonal matrix, where $d_i = \sum_{j=0}^{n} a_{ij}$ for $i = 0, 1, ..., n$. Hence, we define the Laplacian of the weighted graph

$$L = D - A \in \mathbf{R}^{n+1\times n+1}$$

Theorem 5. W.Ren (2005) The Laplacian matrix L of a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right-half plane. In addition, L has exactly one zero eigenvalue if and only if $\mathcal{G}$ has a directed spanning tree. Furthermore, Rank(L)=n if only if L has a simple zero eigenvalue.

We also use the Kronecker product in analyzing the interconnected systems. For that reason, the definition of the Kronecker product and some property are recalled.

Definition 6. Lars (2003): Given the matrices $A \in \mathbf{R}^{n\times m}$ $(A = [a_{i,j}])$ and $B \in \mathbf{R}^{p\times q}$, the Kronecker product of A and B, denoted $A \otimes B$, is the $np \times mq$ matrix

$$A \otimes B = \begin{pmatrix} a_{1,1}B & \dots & a_{1,m}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \dots & a_{n,m}B \end{pmatrix}$$

Throughout this paper, We use $z = (z_1, ..., z_n)^T$ to denote the vector in $\mathbf{R}^n$. Let

$$\zeta_\alpha(z) = (sign(z_1)|z_1|^\alpha, ..., sign(z_n)|z_n|^\alpha)^T$$

3. FINITE-TIME STABILITY ANALYSIS OF INTERCONNECTED SYSTEMS.

Consider the i^{th} interconnected system

$$\dot{x}_i = f_i(t, x_i) + g_i(t, x) \quad i = 1, ..., N \tag{5}$$

where $x_i = [x_{i_1}, ..., x_{i_n}]^T \in \mathbf{R}^n$, and $x = [x_1, ..., x_N]^T$. Suppose that f_i and g_i are continuous with respect to state variables $x_1, ..., x_N$. However, they do not satisfy the Lipschitz condition at the agreement states, which is the least requirement for finite-time stability because Lipschitz continuity can only lead to asymptotical convergence. Moreover, there exists at least one solution (not unique) of differential equations (5) on $[0, \infty)$ for any initial state in a domain interest, and

$$f_i(t, 0) = 0, \quad g_i(t, 0) = 0, \quad \forall i$$

The analysis consists in ignoring the interconnection terms g_i, and we focus only in the system decomposes into N isolated subsystems:

$$\dot{x}_i = f_i(t, x_i) \tag{6}$$

Let consider that each subsystem has an equilibrium point at the origin $x_i = 0$. So, we begin by finding the Lyapunov function establishing finite-time stability of the origin of each isolated subsystem. Assume that this leads to the objective, the positive definite Lyapunov function $V_i(t, x)$ whose derivative along the trajectories of (6) satisfies the inequality (see Theorem 3)

$$\dot{V}_i(t, x) \le -k_i[V_i(t, x)]^{\lambda_i}, \quad \lambda_i \in (0, 1). \tag{7}$$

Proposition 7. Assume that the interconnection term $g_i(t, x)$ satisfies the following inequality

$$\|g_i(t, x)\| \le \sum_{j=1}^{N} \theta_{ij}\psi_j(x_j) \tag{8}$$

for all $t \ge 0$ and for some nonnegative constants θ_{ij}, where $\psi_j : \mathbf{R}^n \to \mathbf{R}$ are positive definite and continuous function. The interconnected system (5) equilibrium is locally finite-time stable if the following statements hold:

i) $f_i(t, .)$ homogeneous of degree $m < 0$.

ii) the system (6) equilibrium is finite-time stable.

Proof.
Let us define $V(t, x)$ as a Lyapunov function that combines the interaction of N isolated subsytems

$$V(t, x) = \sum_{i=1}^{N} c_i V_i(t, x), \quad c_i > 0 \tag{9}$$

The derivative of $V(t, x)$ along the trajectories of (5) is given by

$$\dot{V}(t, x) = \sum_{i=1}^{N} c_i[\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t, x_i)] + \sum_{i=1}^{N} c_i \frac{\partial V_i}{\partial x_i} g_i(t, x)$$

From the existence of a Lyapunov function for the i^{th} isolated subsystem, the first term on the right-hand side is bounded by $-\sum_{i=1}^{N} c_i k_i[V_i(t, x)]^\lambda$. Further, the Lyapunov function V_i is homogeneous of degree $l > \max\{0, -m\}$ and $\sum_{k=1}^{n} \frac{\partial V_i}{\partial x_{i_k}} = lV_i$. Then the derivative of V satisfies the inequality

$$\dot{V}(t, x) \le -\sum_{i=1}^{N} c_i k_i[V_i(t, x)]^\lambda + \sum_{i=1}^{N} c_i l V_i \sum_{j=1}^{N} \theta_{ij}\psi_j(x_j)$$

$$\le -\sum_{i=1}^{N} c_i[V_i(t, x)]^\lambda[k_i - \sum_{j=1}^{N} lV_i^{1-\lambda}\theta_{ij}\psi_j(x_j)]$$

Since $1 - \lambda > 0$, $V_i(t, .)$ and $\psi_j(.)$ are continuous functions which takes 0 at the origin, there exists $\delta > 0$ such as $\forall x \in B_\delta(0)$

$$\dot{V} \le -\frac{1}{2}\sum_{i=1}^{N} c_i k_i[V_i(t, x)]^\lambda$$

Recall that for $\xi_1, ..., \xi_N \ge 0$ and for $\lambda \in (0, 1)$, we have

$$(\sum_{i=1}^{N} \xi)^\lambda \le \sum_{i=1}^{N} \xi^\lambda \le N^{1-\lambda}(\sum_{i=1}^{N} \xi)^\lambda$$

Then, it's straightforward to find

$$\dot{V} \le -\frac{1}{2}\min_i\{k_i\}\min_i\{c_i^{1-\lambda}\}[V(t, x)]^\lambda$$

From Theorem.2, the above differential inequality permits to conclude the Lyapunov function $V(t, x)$ reaches zero in finite time. Therefore the equilibrium of system (5) is locally finite-time stable. This ends the proof.

4. FINITE-TIME STABILIZATION OF CONTROLLED MULTI-SYSTEM

In recent years, the coordination problem of multi-agent systems has received a lot of attention from various scientific searchers due to the diversity of applications in various areas such as mobile robots, air traffic control, scheduling of automated highway systems, unmanned air vehicles, autonomous underwater vehicles, sensor networks and satellites.
However, the weakness arising from multi-agent systems is to develop distributed control policies based on local information that enables all agents to reach an agreement on certain quantities of interest.
For cooperative control strategies to be successful, numerous issues must be addressed, including the definition and management of shared information among a group of agents to facilitate the coordination of these agents. In this section, we will propose to solve the finite-time stabilization of interconnected system where each individual kinematic/dynamic subsystem is nonlinear and can integrate drift terms.
Our approach consists to consider each subsystem as finite-time stable with the associated Lyapunov function. Consequently, each nonlinear control law is constructed taking each subsystem properties and should ensure the formation's stability in finite-time.
Based on properties of each subsystem and those of interconnections, sufficient conditions are given for finite-time stabilization of interconnected nonlinear systems. Let consider the dynamic of N subsystems indexed by the set $\mathcal{I} = \{1, ..., N\}$, in matrix form, the i^{th} subsystem is taken as,

$$\dot{x}_i = h_i(x_i) + B(x_i)u_i \tag{10}$$

where $x_i \in \mathbf{R}^n$ and the continuous maps $h_i : \mathbf{R}^n \to \mathbf{R}$ and $\forall 1 \leq j \leq m$. Given $B(x_i)$ is in $M_{n,m}$ and the control $u_i \in \mathbf{R}^{m\times n}$.
Subsystem (10) model the behavior of a large variety of autonomous systems, generally are underactuated. Our aim consist to interact multiple systems via the control input u_i while keeping the finite-time stability of each subsystem. The all system is now defined by

$$\dot{x} = h(x) + (I_N \otimes B(x))u \tag{11}$$

such as I_N is the identity matrix, $x \in \mathbf{R}^{Nn}$, $u \in \mathbf{R}^{Nm}$ and $h(x) = (h_1(x_1), ..., h_N(x_N))^T$.
Interconnection in system (11) is the subject to design the control-input u tacking the Laplacian L related to a proposed graph $\mathcal{G}$ (more details are in section 2). Sufficient conditions for finite-time stabilization of (11) are derived in following proposition.

Proposition 8. Under the control input

$$u(t,x) = u^{stf} + u^{inter} \tag{12}$$

where the control vector u^{stf} assumed to be finite-time stable for (11), and the finite-time interconnection with respect to $\mathcal{G}$,

$$u^{inter} = -[L \otimes I_m][I_N \otimes C]\zeta_\alpha(x)$$

C is a matrix in $M_{n,m}$ such that BC is positive semi-definite, then the interconnected system (11) origin is finite-time stable.

Proof. System (10) under (12) can be rewritten as:

$$\begin{aligned}\dot{x}_i &= h_i(x_i) + Bu_i^{stf} + Bu_i^{inter} \\ &= h_i(x_i) + Bu_i^{stf} - \sum_{j=1}^{N} a_{ij} BC[\zeta_\alpha(x_i) - \zeta_\alpha(x_j)]\end{aligned}$$

Note that i^{th} subsystem takes the same form as in (5), with:

$$f_i(t,x) = h_i(x_i) + Bu^{stf} - \sum_{j=1}^{N} a_{ij} BC\zeta_\alpha(x_i) \tag{13}$$

$$g_i(t,x) = \sum_{j=1}^{N} a_{ij} BC\zeta_\alpha(x_j) \tag{14}$$

Suppose that under control u^{stf} each closed loop subsystems is homogenous of degree negative with respect to a dilatation Δ_ε, consequently is finite-time stable. Let consider

$$e_i(t,x_i) = h_i(x_i) + Bu_i^{stf}$$

Then the function $f_i(t,x)$ is as

$$f_i(t,x_i) = e_i(t,x_i) + \delta_i BC\zeta_\alpha(x_i) \tag{15}$$

where $\delta_i = \sum_{j=1}^{N} a_{ij}$.
To study the finite-time stability of $\dot{x}_i = e_i(t,x_i) + \delta_i BC\zeta_\alpha(x_i)$ we use the same technique in proof 1, when the unperturbed system equilibrium $\dot{x}_i = e_i(t,x_i)$ is finite-time stable and the perturbation term $\delta_i BC\zeta_\alpha(x_i)$ is continuous and definite function then we conclude that the system (15) equilibrium is finite-time stable. For the homogeneous proprieties, it is obvious to verify the local homogeneous of the same degree with respect to the same dilation of $e_i(t,.)$. Thus, $\dot{x}_i = f_i(t,x_i)$ is finite-time stable and locally homogeneous.

Now, from the function $g_i(t,x) = \sum_{j=1}^{N} a_{ij} BC\zeta_\alpha(x_j)$, we obtain

$$\|g_i(t,x)\| \leq \sum_{j=1}^{N} a_{ij}\psi_j(x_j)$$

where $\psi_j(x_j) = \|BC\zeta_\alpha(x_j)\|$, which is continuous and positive definite.
Thus, system (11) is finite-time stable. This ends the proof.

Remark 9. As we will see for the multi-unicycle case (section.5), subsystem (10) can be reduced to driftless one. The finite-time stability and interconnection of driftless multi-subsystem can be easily achieved.

Remark 10. In u^{inter}, while replacing $\zeta_\alpha(x_i)$ (respectively $\zeta_\alpha(x_j)$) by x_i, further if u^{stf} leads to an asymptotic behavior of each subsystem, it implies the asymptotic stability of interconnected systems which are studied by Lars (2003). The consensus realization is also a part of our analysis related to the output $y = C\zeta_\alpha$.

5. FINITE-TIME TRACKING OF MULTI-UNICYCLE

Consider a set of N subsystems where the i^{th} ($i \in \{1, ..., N\}$) is in the form:

$$\dot{q}_i = P(q_i)u_i \tag{16}$$

with $\forall i \in \{1,...,N\}$ $P(q_i) = \begin{pmatrix} \cos(\theta_i) & 0 \\ \sin(\theta_i) & 0 \\ 0 & 1 \end{pmatrix}$ and $u_i = \begin{pmatrix} u_{1,i} \\ u_{2,i} \end{pmatrix}$

The system of errors is given by (for more details see P.Morin (2001), Zoghlami (2013))

$$\begin{aligned} \dot{z}_{1,i} &= u_{2,r} z_{2,i} + w_{1,i} \\ \dot{z}_{2,i} &= u_{2,r} z_{1,i} + u_{1,r} z_{3,i} + w_1 z_{3,i} \\ \dot{z}_{3,i} &= w_{2,i} \end{aligned} \quad (17)$$

We propose to write

$$w_i = w_i^{stf} + w_i^{inter}$$

where $w_i = (w_{1,i}, w_{2,i})^T$.

Proposition 11. Let consider a Laplacian matrix L that describes the graph connection between all unicycles, and C is in $M_{3,2}$ such that assumptions in Proposition 8 are verified. The origin of system (17) is finite-time stable under the following tracking control laws

$$\begin{aligned} w_{1,i}^{stf} &= -|u_{1,r}|(sign(z_{1,i})|z_{1,i}|^{\alpha} + z_{2,i} z_{3,i}) \\ w_{2,i}^{stf} &= -|u_{1,r}|[z_{2,i} - sign(z_{1,i})|z_{1,i}|^{\alpha} \\ &+ (z_{1,i} z_{2,i} + sign(z_{1,i})|z_{1,i}|^{\alpha} z_{2,i} \\ &+ z_{2,i}^2 z_{3,i} - \frac{1}{\arctan(\varepsilon)} sign(z_{3,i})|z_{2,i}|^{\alpha+1})] \end{aligned}$$

and $w_i^{inter} = -\sum_{j=1}^{N} a_{ij} C(\zeta_\beta(q_i) - \zeta_\beta(q_j))$

$A = [a_{ij}]$ is the adjacency matrix, deduced from L.

We now present a set of $N = 4$ unicycles and for one unicycle the observation matrix is C and following to the graph $\mathcal{G}$, given by Fig.1, the Laplacian matrix is L

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad L = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

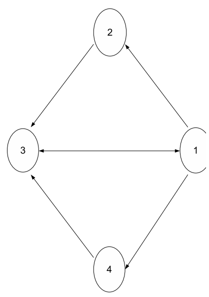

Fig. 1. $\mathcal{G}$ for a system with 4 unicycles.

The control parameter is taken $\alpha = \beta = 0.5$. The unicycles initial positions are given by (in meters):

$$(x_1(0), x_2(0), x_3(0), x_4(0)) = (-3, -2, -1, -4)$$
$$(y_1(0), y_2(0), y_3(0), y_4(0)) = (0, 0, 0, 0)$$

and the initial heading angles are such that (in radian)

$$(\theta_1(0), \theta_2(0), \theta_3(0), \theta_4(0)) = (\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{3\pi}{4}, \frac{3\pi}{4})$$

The initial positions of the reference are given by

$$(x_r(0), y_r(0), \theta_r(0)) = (0, 0, 0)$$

The reference in velocities are as

$$u_{1,r} = 1 m.s^{-1}, \quad u_{2,r} = \cos(t) - \frac{1}{4} rad.s^{-1}.$$

The theoretical results of the paper were confirmed through the obtained simulations. Further, behaviors in tracking of the multi-unicycle system are presented by figure Fig.2.

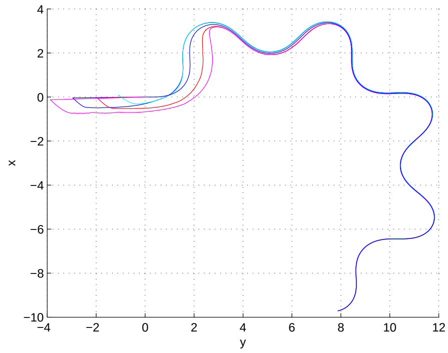

Fig. 2. The phase plot for 4 unicycles

6. CONCLUSION

The finite-time stability conditions for interconnected systems are rigorously established, and the theoretically stability results solve the finite-time stabilization of large variety of controlled multiple autonomous systems with and without drift terms. Also interaction and consensus are obtained in finite-time using the well-known graph theory. Other objectives for multiple systems coordination can be added to the proposed control algorithm. The 4 unicycles tracking results including a finite-time consensus confirm our control approach.

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