Finite-time consensus for controlled dynamical systems in network

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Abstract

The key challenges in controlled dynamical systems in networks are the component heterogeneities, nonlinearities, and the high dimension of the vector of states describing the networked models. Through the existing models especially for autonomous systems treated as controlled first-order ordinary differential equations, nonlinear dynamic models take two main forms that will be addressed in this paper. For each model evolving in networks forming a homogenous or heterogeneous multi-system, protocols integrating sufficient conditions are derived leading to finite-time consensus. Likewise, for the networking topology, we make use of fixed directed and undirected graphs. Finite-time stability theory and Lyapunov methods are useful to prove our approaches. As illustrative examples, the homogeneous multi-unicycle kinematics and the homogenous/heterogenous multi-agent dynamics in networks are detailed.

Index Terms

Finite-time consensus, controlled nonlinear dynamics, multi-system in networks.

I. INTRODUCTION

Networked dynamical systems is an emergent scientific field that brings together multi-agent systems and multi-dynamic analysis. To form a collective task or achieve a global behavior, our main objective is to take into account controlled dynamics in interactions and behavior of traditional networks in multi-agent theory. However, the weakness arising from multi-agent systems is to develop distributed control policies based on local information that enables all

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agents to reach an agreement on certain quantities of interest. This type of collective movement is called *consensus*. Thus, the problem of consensus plays a central role in the study of multi-agent systems. Under the control of a group of mobile agents, it is desirable to obtain a coherent collective movement: the agents are close to each other, avoid collisions and adopt a common direction [9].

The consensus problem was initially used in computer science. In recent years this paradigm has introduced in multi-agent systems witnessed dramatic advances of various distributed strategies that achieve agreements. In [5] the authors proposed a simple but interesting discrete-time model of finite agents all moving in the plane. The proposed model used for the computer animation industry. Each agent's motion is updated using a local rule based on its own state and the states of its neighbors. Jadbabaie *et al.* [6] provided a theoretical explanation of the consensus property of the Vicsek model by using graph theory and nonnegative matrix theory. For this model each agent's set of neighbors changes with time as system evolves. Olfati-Saber and Murray [7] suggested a typical continuous-time model. In this model the concepts of solvability of consensus problems and consensus protocols were first introduced. The authors used a directed graph to model the communication topology among agents and studied three consensus problems, namely, directed networks with fixed topology, directed networks with switching topology, and undirected networks with communication time-delays and fixed topology. Ren and Beard [8] extended the results of Jadbabaie [6] and Olfati-Saber [7] presented mathematically weaker conditions for state consensus under dynamically changing directed interaction topology.

However, finite-time consensus, is one of interesting research problem in consensus, refers to the agreement of a group of agents on a common state in finite time. Finite-time consensus firstly was studied by Cortes [10], where a non-smooth consensus algorithm is proposed. In the same filed [11], and in [15] authors proposed a continuous nonlinear consensus algorithm to guarantee the finite-time stability under an undirected fixed interaction graph. Wang and Xiao in [14] suggest an improvement to the proposed algorithm proposed in [11]. The new algorithm proposed in [14] is able to guarantee finite-time consensus under an undirected switching interaction and a directed fixed interaction graph when each strongly connected component of the topology is detail-balanced.

In [17], the authors study finite-time consensus for second order dynamics with inherent nonlinear dynamics under an undirected fixed interaction graph. Recently, various finite-time stabilizing control laws have been proposed using continuous state feedback and output feedback controllers Bhat *et al.* [3]. Furthermore, the finite-time control design has been extended to n^{th} order systems with both parametric and dynamic uncertainties [2]. Although the finite-time design is generally more difficult than the asymptotically stabilizing control due to the lack of effective analysis tools. Also, the non-smooth finite-time control synthesis can improve the system behaviors in some aspects like high-speed, control accuracy, and disturbance- rejection. Therefore, it is not surprising that finite-time control ideas have been applied to multi-agent systems with first-order agent dynamics using gradient flow and Lyapunov function [10]-[12].

What motivated the paper is the finite-time consensus of controlled nonlinear dynamics in networks where two types of dynamic models are emphasized. These models are with/without drift terms, commonly used for autonomous systems modeling. The proposed protocols solve the problem of homogenous/heteregoneous finite-time consensus under fixed and directed/undirected topology. Results from finite time stability theory, graph theory, and Lyapunov techniques are recalled and used to prove our approaches.

The paper is organized as follows. First, preliminaries and problem formulation are shown in Section II. Then we focus on the finite-time consensus of networked driftless systems in section III. Sufficient condition to finite-time consensus of networked drift systems are given in section IV. In section V, we present result for finite-time stabilization of networked driftless/drift nonlinear systems. The paper is ended by concluding remarks.

II. Preliminaries and problem formulation

A. Notations

Throughout this paper, we use $\mathbb{R}$ to denote the set of real number. $\mathbb{R}^n$ is the n-dimensional real vector space and $\|.\|$ denotes the Euclidian norm. $\mathbb{R}^{n\times n}$ is the set of $n \times n$ matrices. $diag\{m_1, m_2, ..., m_n\}$ denotes a $n \times n$ diagonal matrix. $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. The symbol $\otimes$ is the Kronecker product of matrices. We use $sgn(.)$ to denote the signum function. For a scalar x, note that $\varphi_\alpha(x) = sgn(x)|x|^\alpha$. We use $x^i = (x^i_1, ..., x^i_n)^T \in \mathbb{R}^n$, $\mathbf{x} = (x^1, ..., x^N)^T$ to denote the vector in $\mathbb{R}^{nN}$. Let $\phi_\alpha(x^i) = (\varphi_\alpha(x^i_1), ..., \varphi_\alpha(x^i_n))^T$ with $\phi_\alpha(\mathbf{x}) = (\phi_\alpha(x^i), ..., \phi_\alpha(x^N))^T$. For $z = (z_1, ..., z_n)$ vector in $\mathbb{R}^n$, $\delta(z) = [|z_1|, ..., |z_n|]^T$ and $\delta^\gamma(z) = [|z_1|^\gamma, ..., |z_n|^\gamma]^T$ for $\gamma > 0$. Let $\mathbf{1}_n = (1, ..., 1)^T$. The exponent T is the transpose.

B. Graph theory

In this subsection, we introduce some basic concepts in algebraic graph theory for multi-agent networks (More notions in graph theory are in [4]). Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a directed graph, where $\mathcal{V} = \{1, 2, ..., n\}$ is the set of nodes, node i represents the ith agent, $\mathcal{E}$ is the set of edges, and an edge in $\mathcal{G}$ is denoted by an ordered pair (i, j). $(i, j) \in \mathcal{E}$ if and only if the ith agent can send information to the jth agent directly.

$A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements, where $a_{ij} > 0$ if there is an edge between the ith agent and jth agent and $a_{ij} = 0$ otherwise. Moreover, if $A^T = A$, then $\mathcal{G}$ is also called an undirected graph. In this paper, we will refer to graphs whose weights take values in the set$\{0, 1\}$ as binary and those graphs whose adjacency matrices are symmetric as symmetric. Let $D = diag\{d_1, ..., d_n\} \in \mathbb{R}^{n\times n}$ be a diagonal matrix, where $d_i = \sum_{j=1}^{n} a_{ij}$ for $i = 0, 1, ..., n$. Hence, we define the Laplacian of the weighted graph

$$L = D - A \in \mathbb{R}^{n\times n}.$$

The undirected graph is called connected if there is a path between any two vertices of the graph. Directed graph is strongly connected if between every distinct pair (i, j) in $\mathcal{G}$, there is a path that begins at i and ends at j.

We say that a directed graph has a spanning tree if a subset of the edges forms a spanning tree (where a spanning tree of $\mathcal{G}$ is a directed tree that is spanning subgraph of $\mathcal{G}$).

Note that time varying network topologies are not considered in this paper.

C. Some useful lemmas

In order to establish our main results, we need to recall the following Lemmas.

Lemma II.1. *[3]. Consider the system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, $f(0) = 0$, $\boldsymbol{x} \in \mathbb{R}^n$, there exist a positive definite continuous function $V(\boldsymbol{x}) : U \subset \mathbb{R}^n \to \mathbb{R}$, real numbers $c > 0$ and $\beta \in]0, 1[$, and an open neighborhood $U_0 \subset U$ of the origin such that $\dot{V} + c(V(\boldsymbol{x}))^\beta \leqslant 0$, $\boldsymbol{x} \in U_0 \backslash \{0\}$. Then $V(\boldsymbol{x})$ converges to zero in finite time. In addition, the finite settling time T_* satisfies $T_* \leqslant \dfrac{V(\boldsymbol{x}(0))^{1-\beta}}{c(1-\beta)}$.*

Lemma II.2. *[7].*

(i) *If $\mathcal{G}$ has a spanning tree, then eigenvalue 0 is algebraically simple and all other eigenvalues are with positive real part.*

(ii) *If $\mathcal{G}$ is strongly connected, then there exists a positive column vector $w \in \mathbb{R}^n$ such that $w^T L = 0$*

Lemma II.3. *[14] Suppose $\mathcal{G}$ is strongly connected, and let $w > 0$ such that $w^T L = 0$. Then $diag(w)L + L^T diag(w)$ is the Laplacien matrix of the undirected weighted graph $\mathcal{G}(diag(w)L + L^T diag(w))$. And therefore it is semi-positive definite, 0 is its algebraically simple eigenvalue and **1** is the associated eigenvector.*

Lemma II.4. *[7]. For a connected undirected graph $\mathcal{G}$, the Laplacian matrix L of $\mathcal{G}$ has the following properties, $\boldsymbol{x}^T L \boldsymbol{x} = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x_i - x_j)^2$, which implies that L is positive semi-definite. 0 is a simple eigenvalue of L and **1** is the associated eigenvector. Assume that the eigenvalues of L are denoted by $0, \lambda_2, ..., \lambda_n$ satisfying $0 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_n$. Then the second smallest eigenvalue satisfies $\lambda_2 > 0$. Furthermore, if $\boldsymbol{1}^T \boldsymbol{x} = 0$, then $\boldsymbol{x}^T L \boldsymbol{x} \geqslant \lambda_2 \boldsymbol{x}^T \boldsymbol{x}$.*

Lemma II.5. *[18]. Let $x_1, x_2, ..., x_n \geqslant 0$ and $0 < p \leqslant 1$.*
Then $(\sum_{i=1}^{n} x_i)^p \leqslant \sum_{i=1}^{n} x_i^p \leqslant n^{1-p}(\sum_{i=1}^{n} x_i)^p$.

D. Problem statements

There are two main controlled nonlinear dynamic models issued from autonomous systems modeling. Hence, the key challenges are the complexity of these models when they are considered in networks, likewise, the size of each system's vector of states is large, and increases greatly with the considered number of agents. Further, most of the autonomous system dynamics are nonlinear and underactuated but modeled with well known first-order ordinary differential equations or equivalently. Such an equation is controlled, and may integrate drift terms which make difficult the stability analysis of networked dynamical systems.

In the following, we propose to solve the finite-time consensus problem of two controlled dynamical systems in network. The first one is based on model (1) which describes a controlled dynamical system without drift term, and it will be referred as Σ_1. The second model is a controlled dynamic system which is represented by (2), which is clearly a controlled dynamic with drift term, and it will be assigned to Σ_2.

Consider a multi-agent group as N high-dimensional dynamical systems where each agent's behavior is described by ($\forall i \in \mathcal{I} = \{1, ..., N\}$)

$$\Sigma_1 : \quad \dot{x}^i = B(x^i)u^i \tag{1}$$

$$\Sigma_2 : \quad \dot{x}^i = f^i(x^i) + B(x^i)u^i \tag{2}$$

where $x^i \in \mathbb{R}^n$, and for $1 \leqslant i \leqslant N$ $x^i = [x_1^i, x_2^i, ..., x_n^i]^T$, $B(x^i) \in \mathbb{R}^{n\times m}$, the continuous maps $f^i : \mathbb{R}^n \to \mathbb{R}^n$, $u^i \in \mathbb{R}^m$ is the control-input which depends only on the neighbor states. The matrix $B(x^i) = [b_{kl}]$ for $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant m$.

In the paper, a networked dynamical system based on (1) is referred to *multi*-Σ_1. However, if the networked dynamical system is referred to (2), then it is assigned to *multi*-Σ_2. Note that the multi-Σ_1 is assumed to be homogeneous multi-system with respect to $B(.)$ structure, and the multi-Σ_2 must be homogenous if $f^i(.)$ structure doesn't change, but the result of the paper is extended to the heterogenous case where $f^i(.)$ can be different for each agent.

Définition II.6. *For multi-Σ_1 or multi-Σ_2, given a protocol u^i, a consensus problem is solved in finite time, if for any Σ_1, Σ_2 state initial conditions there exists a finite-time T_* such that*

$$\lim_{t\to T_*} \|x^i(t) - x^j(t)\| = 0 \tag{3}$$

for any $i, j \in \mathcal{I}$.

Throughout the paper, a consensus protocol candidate that expected to solve finite-time consensus for multi-Σ_1 and multi-Σ_2 is given by ($i \in \mathcal{I}$)

$$u^i = -C(x^i)\phi_\alpha(\sum_{j=1}^{N} a_{ij}(x^i - x^j)) \tag{4}$$

where $C(x^i) \in \mathbb{R}^{m\times n}$, $\alpha \in]0, 1[$, and a_{ij} are the adjacent elements related to the specified $\mathcal{G}$.

Assumption II.7. *Let $\tilde{B} = B(x^i)C(x^i)$ with $\tilde{B} = [\tilde{b}_{mk}]_{m,k}$ for $1 \leqslant m, k \leqslant n$. We assume that there exists $C(x^i)$ such that $\tilde{B}$ is a positive semi-definite matrix.*

Assumption II.8. *A Σ_2 drift term f^i in (2) satisfies the following inequality ($\forall i \in \mathcal{I}$)*

$$\|\sum_{j=1}^{N} a_{ij}(f^i(x^i) - f^j(x^j))\| \leqslant \mu\|\sum_{j=1}^{N} a_{ij}(x^i - x^j)\| \tag{5}$$

with μ is a positive constant.

III. Finite-time consensus in multi-Σ_1 dynamical system

In the following, the networked dynamical systems is subjective to multi-Σ_1 analysis under the protocol (4). Further, as networked topology we will distinguish a directed and an undirected graphs. The reader may find more details about the graph theory in [4]. To solve a finite-time consensus, our approach is addressed in Proposition III.1 for a given directed graph, and in Proposition III.2 for the undirected graph case.

Proposition III.1. *For a given fixed directed graph $\mathcal{G}$ strongly connected, the protocol (4) associated to multi-Σ_1 solves a finite-time consensus problem.*

Proof. For $\mathbf{x} = (x^1, ...x^N)^T$ and $\mathbf{u} = (u^1, ..., u^N)^T$, the multi-$\Sigma_1$ is defined by

$$\dot{\mathbf{x}} = I_N \otimes B(x^i)\mathbf{u} \tag{6}$$

One starts the analysis by an adequate change of variable, where for $i \in \mathcal{I}$,

$$y^i = \sum_{j=1}^{N} a_{ij}(x^i - x^j) \tag{7}$$

Therefore,

$$\dot{x}^i = \tilde{B}\phi_\alpha(y^i) \tag{8}$$

Let us rewrite the protocol (4) as $u^i = -C(x^i)\phi_\alpha(y^i)$, further let $\mathbf{y} = (y_1, ..., y_N)^T$, then in a compact form

$$\mathbf{u} = -(I_N \otimes C(x^i))\phi_\alpha(\mathbf{y}) \tag{9}$$

From (7), we have

$$\mathbf{y} = (L \otimes I_n)\mathbf{x} \tag{10}$$

At this stage, we are able to introduce the protocol (4) into the multi-Σ_1, and this after the time derivative of (10). This leads to

$$\begin{aligned}\dot{\mathbf{y}} &= (L \otimes I_n)\dot{\mathbf{x}} \\ &= -(L \otimes I_n)(I_N \otimes B(x^i))(I_N \otimes C(x^i))\phi_\alpha(\mathbf{y}) \\ &= -(L \otimes \tilde{B})\phi_\alpha(\mathbf{y})\end{aligned} \tag{11}$$

where in the last written we use the Kronecker product properties (see [1]). The goal is to prove that $\mathbf{y}$ reaches zero in finite time. As $\mathcal{G}$ is strongly connected, there exists a vector $\mathbf{w} = [w^1, w^2, ..., w^N]^T \in \mathbb{R}^{n\times N}$, such that $\mathbf{w}^T L(A) = 0$ (by lemmaII.3), and where for $1 \leqslant i \leqslant N$, $w^i = [w_1^i, w_2^i, ..., w_n^i]^T$. Taking the Lyapunov function candidate

$$V(\mathbf{y}) = \frac{1}{1+\alpha}\sum_{i=1}^{N} < w^i, \delta^{1+\alpha}(y^i) > \tag{12}$$

where $< ., . >$ denotes the scalar product. A detailed form is given by

$$V(\mathbf{y}) = \frac{1}{1+\alpha}\sum_{i=1}^{N}\sum_{k=1}^{n} w_k^i |y_k^i|^{1+\alpha}$$

Evaluating V along the transformed vector field solutions, using (11) and (7)-(8), we obtain

$$\begin{aligned}\dot{V}(\mathbf{y}) &= \sum_{i=1}^{N}\sum_{k=1}^{n} w_k^i \varphi_\alpha(y_k^i)\frac{dy_k^i}{dt} \\ &= \sum_{i=1}^{N}\sum_{k=1}^{n} w_k^i \varphi_\alpha(y_k^i)(\sum_{j=1}^{N} a_{ij}(\dot{x}_k^i - \dot{x}_k^j)) \\ &= \sum_{i=1}^{N}\sum_{k=1}^{n} w_k^i \varphi_\alpha(y_k^i)(\sum_{j=1}^{N}\sum_{m=1}^{n} a_{ij}[\tilde{b}_{km}\varphi_\alpha(y_m^i) - \tilde{b}_{km}\varphi_\alpha])\end{aligned} \tag{13}$$

After having processed the last equality in matrix form, it is straightforward to prove that

$$\dot{V}(\mathbf{y}) = -\phi_\alpha^T(\mathbf{y})(I_N \otimes diag(w))(L \otimes \tilde{B})\phi_\alpha(\mathbf{y})$$

Now, let $E \triangleq \frac{1}{2}((diag(w)L \otimes \tilde{B}) + (L \otimes \tilde{B}diag(w))^T$, which means that

$$\dot{V} = -\phi_\alpha^T(\mathbf{y})E\phi_\alpha(\mathbf{y}) \tag{14}$$

Introduce

$$\Omega = \{\mathbf{z} \in \mathbb{R}^{nN} : \mathbf{z}^T\mathbf{z} = 1 \ \text{ and } \ z = \phi_\alpha(\vartheta) \ \text{ for } \ \vartheta \perp w\}$$

which is obviously a compact set, and as the function $z^T E z$ is continuous in Ω, then a nonzero minimum exists $\min_{z\in\Omega} z^T E z \neq 0$. Moreover, E is the Laplacian matrix of a undirected weighted graph $\mathcal{G}(E)$, and it is positive semi-definite (Lemma II.3). Then, $\min_{z\in\Omega} z^T E z > 0$.

Let $K_1 = \min_{z\in\Omega} z^T E z > 0$, as $\frac{\phi_\alpha(\mathbf{y})}{\sqrt{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})}} \in \Omega$, then

$$\frac{\phi_\alpha^T(\mathbf{y})E\phi_\alpha(\mathbf{y})}{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})} = \frac{\phi_\alpha^T(\mathbf{y})}{\sqrt{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})}} E \frac{\phi_\alpha(\mathbf{y})}{\sqrt{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})}} \geqslant K_1$$

The goal is to prove that the derivative of V satisfies $\dot{V} \leqslant -cV^{\beta}$ (by Lemma II.1). Using (14), we obtain

$$\begin{aligned}\dot{V} &= \frac{\phi_\alpha^T(\mathbf{y})E\phi_\alpha(\mathbf{y})}{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})}\frac{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})}{V^\beta}V^\beta \\ &\leqslant -K_1\frac{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})}{V^\beta}V^\beta\end{aligned}$$

From the fact that

$$\begin{aligned}\frac{\phi_\alpha^T(\mathbf{y})\phi_\alpha(\mathbf{y})}{V^\beta} &= \frac{\displaystyle\sum_{i=1}^{N}\sum_{k=1}^{n}|y_k^i|^{2\alpha}}{(\displaystyle\sum_{i=1}^{N}\sum_{k=1}^{n}\frac{w_k^i}{\alpha+1}|y_k^i|^{1+\alpha})^\beta} \\ &\geqslant \frac{\displaystyle\sum_{i=1}^{N}\sum_{k=1}^{n}|y_k^i|^{2\alpha}}{\displaystyle\sum_{i=1}^{N}\sum_{k=1}^{n}(\frac{w_k^i}{\alpha+1})^\beta|y_k^i|^{(1+\alpha)\beta}} \quad (\text{Lemma II.5})\end{aligned}$$

if we choose $\beta = \frac{2\alpha}{1+\alpha}$, and let $k_2 = \max_i \max_k (\frac{w_k^i}{\alpha+1})^\beta$. Obviously $k_2 > 0$. Finally, we can prove that there exists $c = \frac{K_1}{K_2} > 0$, meaning that

$$\dot{V}(\mathbf{y}) \leqslant -c(V(\mathbf{y}))^{\frac{2\alpha}{1+\alpha}} \tag{15}$$

Thus, V will reach zero in finite time $T_*(\mathbf{y}(0)) = \dfrac{(\alpha+1)}{(1-\alpha)c}V(\mathbf{y}(0))^{\frac{1-\alpha}{\alpha+1}}$ (by Lemma II.1). Consequently, the proposed protocol (4) applied to multi-Σ_1 solves a finite-time consensus problem in the sense of (3). This ends the proof.

■

Proposition III.2. *For a given fixed undirected and connected graph $\mathcal{G}$, the protocol (4) associated to multi-Σ_1 solves a consensus problem in finite time.*

Proof. Recall that the multi-Σ_1 is defined by

$$\dot{\mathbf{x}} = I_N \otimes B(x_i)\mathbf{u} \tag{16}$$

with $\mathbf{x} = (x^1, ...x^N)^T$ and $\mathbf{u} = (u^1, ..., u^N)^T$, One considers y^i as in (7), $\mathbf{y}$ as in (10), and $\dot{\mathbf{y}}$ is as in (11). The goal is also to prove that $\mathbf{y}$ reaches zero in finite time. Taking the Lyapunov function $V : \mathbb{R}^{Nn} \to \mathbb{R}_+$ such that $\forall\ \mathbf{y} \in \mathbb{R}^{Nn}$

$$V(\mathbf{y}) = \frac{1}{1+\alpha}\mathbf{y}^T\phi_\alpha(\mathbf{y}) \tag{17}$$

which is positive definite with respect to $\mathbf{y}$, and consider the time derivative of V along the trajectories of (11), we get

$$\begin{aligned}\dot{V}(\mathbf{y}) &= \phi_\alpha^T(\mathbf{y})\frac{d\mathbf{y}}{dt}\\ &= -\phi_\alpha^T(\mathbf{y})(L \otimes \tilde{B})\phi_\alpha(\mathbf{y})\end{aligned} \tag{18}$$

Let

$$D(x^i) = \begin{bmatrix} 0_n & & & \\ & \gamma_2(x^i) & & \\ & & \ddots & \\ & & & \gamma_N(x^i) \end{bmatrix}$$

where $0_n = diag\{0, ..., 0\} \in \mathbb{R}^{n\times n}$, and $\forall j = 2, ..., N$ $\gamma_j(x^i) = \lambda_j(L)\varrho_n(x^i)$ with $\varrho_n(x^i) = diag\{0, \mu_2(x^i), ..., \mu_n(x^i)\} \in \mathbb{R}^{n\times n}$, and where $\mu_2(x^i), ..., \mu_n(x^i)$ are the eigenvalues of the matrix $\tilde{B}$ given in increasing order. $\lambda_j(L)$ denotes the j^{th} eigenvalue of L. Let them be $\lambda_2(L), ..., \lambda_N(L)$ in increasing order. Since $\mathcal{G}$ is connected (by Lemma II.2) $\lambda_2(L) > 0$. Therefore, $\forall x^i$ we have $\lambda_2\mu_2(x^i) > 0$.

Further, since $L \otimes \tilde{B} \in \mathbb{R}^{Nn\times Nn}$ is symmetric matrix, then there exist an orthogonal matrix $P \in \mathbb{R}^{Nn\times Nn}$ such that $L \otimes \tilde{B} = P^T D(x^i)P$. Let $\mathbf{z}_\alpha = P\phi_\alpha(\mathbf{y})$, thus

$$\begin{aligned}\dot{V} &= -\mathbf{z}_\alpha^T D\mathbf{z}_\alpha\\ &\leqslant -\lambda_2\mu_1(x^i)\|\mathbf{z}_\alpha\|^2\\ &\leqslant -\lambda_2\mu_1(x^i)\|\phi_\alpha(\mathbf{y})\|^2\end{aligned} \tag{19}$$

with $\lambda_2\mu_1(x^i) = \min\limits_{\mathbf{z}_\alpha \perp \mathbf{1}_{Nn}} \dfrac{\mathbf{z}_\alpha^T D\mathbf{z}_\alpha}{\mathbf{z}_\alpha^T\mathbf{z}_\alpha}$.

Let $k = \min\limits_{x^i\in\mathbb{R}^N} \lambda_2\mu_1(x^i) > 0$ and $\mathbf{y} = \mathbf{1}_N \otimes y^i = (\tilde{y}_1, ..., \tilde{y}_{Nn})^T$ consequently,

$$\begin{aligned}\dot{V} &\leqslant -k\sum_{i=1}^{Nn}|\varphi_\alpha(\tilde{y}_i)|^2\\ &\leqslant -k\sum_{i=1}^{Nn}|\tilde{y}_i|^{2\alpha}\\ &\leqslant -k(\sum_{i=1}^{Nn}|\tilde{y}_i|^{\alpha+1})^{\frac{2\alpha}{\alpha+1}} \qquad \text{(by Lemma II.5)} \qquad (20)\end{aligned}$$

Then

$$\dot{V} \leqslant -k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}V^{\frac{2\alpha}{\alpha+1}} \qquad (21)$$

Since $0 < \frac{2\alpha}{\alpha+1} < 1$ and $k(\alpha+1)^{\frac{2\alpha}{\alpha+1}} > 0$, and by Lemma II.1, the above differential equation gives that V reaches zero in finite time $\dfrac{(\alpha+1)V(y(0))^{\frac{1-\alpha}{\alpha+1}}}{(1-\alpha)k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}}$. Therefore, based on (1), the multi-Σ_1 under the protocol (4) leads to a finite-time consensus. This ends the proof.

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Remarque III.3. *For a given fixed directed/undirected graph $\mathcal{G}$, from inequalities (15) and (21) if we take $\alpha = 1$, then the finite-time consensus in multi-Σ_1 becomes an asymptotically consensus.*

Remarque III.4. *The protocol (4) is similar to [14] by taking $B = C = 1$ which was applied to multi-particle system $\dot{x}_i = u_i$ in network.*

IV. Finite-time consensus in multi-Σ_2 dynamical system

Let us recall that a multi-Σ_2 dynamical system is a networked dynamical systems where each system's dynamic model is given by (2). The consensus protocol is as given in (4). Note that various autonomous systems are modeled by (2), and Assumption II.8 can be easily verified. In the following, the consensus problem is analyzed for the directed and undirected graphs, considered has a spanning tree and strongly connected. Recall that for a given graph $\mathcal{G}$, the purpose is to prove $\|x^i(t) - x^j(t)\| \to 0$ in finite time $\forall i, j = 1, ..., N$, while u^i is as in (2).

Proposition IV.1. *If the graph $\mathcal{G}$ has a spanning tree and strongly connected and the drift term satisfies the inequality (5), then the multi-Σ_2 based on (2) with the protocol (4) realize a homogenous/heterogenous consensus in finite-time.*

Proof. Using the change of variable given by (7), we have

$$\dot{y}^i = \sum_{j=1}^{N} a_{ij}(f^i(x^i) - f^j(x^j)) + \sum_{j=1}^{N} a_{ij}[B(x^i)u^i - B(x^j)u^j] \tag{22}$$

For $\mathbf{y} = (y^1, ..., y^N)^T$, $f(\mathbf{x}) = (f^1(x^1), ..., f^N(x^N))^T$ and using (10), the multi-Σ_2 is given by

$$\dot{\mathbf{y}} = (L \otimes I_n)f(\mathbf{x}) - (L \otimes \tilde{B})\phi_\alpha(\mathbf{y}) \tag{23}$$

From inequality (5), we have

$$\|(L \otimes I_n)f(\mathbf{x})\| \leqslant c\|(L \otimes I_n)\mathbf{x}\| = c\|\mathbf{y}\| \tag{24}$$

With respect to the Lyapunov function (12), the time derivative of $V(\mathbf{y})$ along the networked system trajectories (23), we may write

$$\begin{aligned}
\dot{V}(\mathbf{y}) &= \sum_{i=1}^{N}\sum_{k=1}^{n} w_k^i \varphi_\alpha(y_k^i)\frac{dy_k^i}{dt} \\
&= \sum_{i=1}^{N}\sum_{k=1}^{n} w_k^i \varphi_\alpha(y_k^i)(\sum_{j=1}^{N} a_{ij}(\dot{x}_k^i - \dot{x}_k^j)) \\
&= \sum_{i=1}^{N}\sum_{k=1}^{n} w_k^i \varphi_\alpha(y_k^i)(\sum_{j=1}^{N} a_{ij}(f_k^i(x^i) - f_k^j(x^j))) + \dot{V}_{/(1)}
\end{aligned}$$

where $\dot{V}_{/(1)}$ is the derivative of the Lyapunov function with respect to the driftless system (1), given by the previous section and satisfies the inequality (15). Now, using the Assumption II.8 and the equality (13), we obtain

$$\begin{aligned}
\dot{V}(\mathbf{y}) &\leqslant \mu \sum_{i=1}^{N}\sum_{k=1}^{n} w_k^i sgn(y_k^i)|y_k^i|^\alpha (\sum_{j=1}^{N} a_{ij}(x^i - x^j)) - c(V(\mathbf{y}))^{\frac{2\alpha}{1+\alpha}} \\
&\leqslant \mu \sum_{i=1}^{N}\sum_{k=1}^{n} w_k^i |y_k^i|^{\alpha+1} - c(V(\mathbf{y}))^{\frac{2\alpha}{1+\alpha}} \\
&\leqslant \mu(1+\alpha)V(\mathbf{y}) - c(V(\mathbf{y}))^{\frac{2\alpha}{1+\alpha}} \\
&\leqslant -V^{\frac{2\alpha}{\alpha+1}}[c - \mu(1+\alpha)V^{\frac{1-\alpha}{1+\alpha}}]
\end{aligned} \tag{25}$$

Since $\frac{1-\alpha}{1+\alpha} > 0$ and V is continuous function which takes 0 at the origin ($\mathbf{y} = \mathbf{0}$), there exists an open neighborhood Ω of the origin that permits to write

$$\dot{V}(\mathbf{y}) \leqslant -\frac{c}{2}[V(\mathbf{y})]^{\frac{2\alpha}{\alpha+1}} \tag{26}$$

by Lemma II.2, V reaches zero at an estimated finite time

$$T_*(y(0)) = \frac{2(\alpha+1)}{c(1-\alpha)} V(y(0))^{\frac{1-\alpha}{\alpha+1}}$$

Therefore the multi-Σ_2 based on (2) and the protocol (4) lead to a homogenous/hetrogenous finite-time consensus, as $f^i(x^i)$ may (not) be identical in (22). This ends the proof.

■

For the consensus problem of mulit-Σ_2, let us analyze the case of an undirected graph, while the protocol is similar to (4).

Proposition IV.2. *Given an undirected and connected graph $\mathcal{G}$, further we consider that the inequality (5) holds, then the multi-Σ_2 described from (2) with the protocol (4) lead to a homogenous/heterogenous finite-time consensus.*

Proof. From the change of variable given by (7), we have

$$\dot{y}^i = \sum_{j=1}^{N} a_{ij}(f^i(x^i) - f^j(x^j)) + \sum_{j=1}^{N} a_{ij}[B(x^i)u^i - B(x^j)u^j] \tag{27}$$

For $\mathbf{y} = (y^1, ..., y^N)^T$, $f(\mathbf{x}) = (f^1(x^1), ..., f^N(x^N))^T$ and using (10), the multi-Σ_2 is as

$$\dot{\mathbf{y}} = (L \otimes I_n) f(\mathbf{x}) - (L \otimes \tilde{B})\phi_\alpha(\mathbf{y}) \tag{28}$$

While from inequality (5), we have

$$\|(L \otimes I_n) f(\mathbf{x})\| \leqslant \mu \|(L \otimes I_n)\mathbf{x}\| = \mu \|\mathbf{y}\| \tag{29}$$

Using the Lyapunov function (17), and consider the time derivative of $V(\mathbf{y})$ along trajectories of the multi-Σ_2 (28), it leads to

$$\begin{aligned}\dot{V}(\mathbf{y}) &= \phi_\alpha^T(\mathbf{y})(L \otimes I_n) f(\mathbf{x}) - \phi_\alpha^T(\mathbf{y})(L \otimes \tilde{B})\phi_\alpha(\mathbf{y}) \\ &\leqslant \mu |\phi_\alpha^T(\mathbf{y})\mathbf{y}\| - \phi_\alpha^T(\mathbf{y})(L \otimes \tilde{B})\phi_\alpha(\mathbf{y})\end{aligned}$$

Let $\mathbf{y} = \mathbf{1}_N \otimes y^i = (\tilde{y}_1, ..., \tilde{y}_{Nn})^T$, consequently from (20) we get

$$\begin{aligned}\dot{V}(\mathbf{y}) &\leqslant \mu \sum_{i=1}^{Nn} |\tilde{y}_i|^{\alpha+1} - k(\sum_{i=1}^{Nn} |\tilde{y}_i|^{\alpha+1})^{\frac{2\alpha}{\alpha+1}} \\ &\leqslant -V^{\frac{2\alpha}{\alpha+1}}[k(\alpha+1)^{\frac{2\alpha}{\alpha+1}} - \mu V^{\frac{1-\alpha}{1+\alpha}}]\end{aligned} \tag{30}$$

where $k = \min_{x^i \in \mathbb{R}^n} \lambda_2 \mu_1(x^i)$ defined in the proof of Proposition III.1. Since $\frac{1-\alpha}{1+\alpha} > 0$ and V is continuous function which takes 0 at the origin, there exists an open neighborhood Ω of the origin that permits to write

$$\dot{V}(\mathbf{y}) \leqslant -\frac{k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}}{2}[V(\mathbf{y})]^{\frac{2\alpha}{\alpha+1}} \tag{31}$$

From Lemma II.2, V reaches zero at an estimated finite time

$$T_*(y(0)) = \frac{(\alpha+1)V(y(0))^{\frac{1-\alpha}{\alpha+1}}}{2(1-\alpha)k(\alpha+1)^{\frac{2\alpha}{\alpha+1}}}$$

Therefore, the multi-Σ_2 defined from model (2) and the protocol (4) achieve a homogenous/hetrogenous finite-time consensus, as $f^i(x^i)$ may (not) be identical in (22). This ends the proof.

■

Remarque IV.3. *It is straightforward from Proposition IV.2 proofs, if $\alpha = 1$, then the multi-Σ_2 finite-time consensus becomes an asymptotically one.*

V. ILLUSTRATIVE EXAMPLES

Two illustrative examples are considered where the multi-unicycle represents a controlled nonlinear system in network modeled by (1) that follows the theoretical analysis of multi-Σ_1. A multi-agent based on second order dynamics which implies networked multi-model of type (2), and follows the subsequent analysis of multi-Σ_2. Each multi-system associated protocol is deduced from (4) and the results are illustrated by simulations. In figure Fig.1, the directed graph is associated to 3 agents (left) while the undirected graph is applied to 4 agents (right).

A. Multi-unicycle consensus for the rendezvous problem

Consider N wheeled mobile robots where the i^{th} nonholonomic kinematic model is as (unicycle):

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_i \\ w_i \end{pmatrix} \quad i = 1, ..., N \tag{32}$$

where (x_i, y_i, θ_i) denotes the position and the orientation in a inertial frame. The inputs u_i and w_i are the linear and angular velocities, respectively. Let

$$B = \begin{pmatrix} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{pmatrix}$$

Two constructed matrices to C are proposed, and this in order to see the role of the control matrix C in (4). Hence, the control matrix C in (4) is first taken as

$$C_1 = \begin{pmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \end{pmatrix} \quad \text{and second as} \quad C_2 = \begin{pmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 1 \end{pmatrix}$$

It is straightforward to verify the property of BC_1 and BC_2 in Assumption II.7, where their eigenvalues are $\{0, 0, \cos^2\theta_i + \sin^2\theta_i\}$ and $\{0, 1, \cos^2\theta_i + \sin^2\theta_i\}$, respectively. We propose to study the finite-time consensus of multi-unicycle given by (1) as multi-Σ_1 (system without drift term in network). Based on Proposition III.1, the finite-time consensus problem can be solved through the following protocols $((u_i)_{C_1}, (w_i)_{C_1})$ and $((u_i)_{C_2}, (w_i)_{C_2})$ where $(u_i)_{C_1} = (u_i)_{C_2} = u_i$ which are computed following to C_1 and C_2 matrices. Then we obtain,

$$u_i = -\varphi_\alpha(\sum_{j=1}^{N} a_{ij}(x_i - x_j))\cos\theta_i - \varphi_\alpha(\sum_{j=1}^{N} a_{ij}(y_i - y_j))\sin\theta_i \tag{33}$$

$$(w_i)_{C_1} = \varphi_\alpha(\sum_{j=1}^{N} a_{ij}(x_i - x_j))\sin\theta_i - \varphi_\alpha(\sum_{j=1}^{N} a_{ij}(y_i - y_j))\cos\theta_i \tag{34}$$

$$(w_i)_{C_2} = \varphi_\alpha(\sum_{j=1}^{N} a_{ij}(x_i - x_j))\sin\theta_i - \varphi_\alpha(\sum_{j=1}^{N} a_{ij}(y_i - y_j))\cos\theta_i - \varphi_\alpha(\sum_{j=1}^{N} a_{ij}(\theta_i - \theta_j)) \tag{35}$$

Fixed directed graph (Fig. 1a). The following initial conditions for the three unicycles in networks are given by,

$$(x_1, y_1, \theta_1)(t = 0) = (4, 2, \frac{\pi}{4})$$

$$(x_2, y_2, \theta_2)(t=0) = (2, -1, -\frac{\pi}{2})$$
$$(x_3, y_3, \theta_3)(t=0) = (-6, 10, \frac{\pi}{2})$$

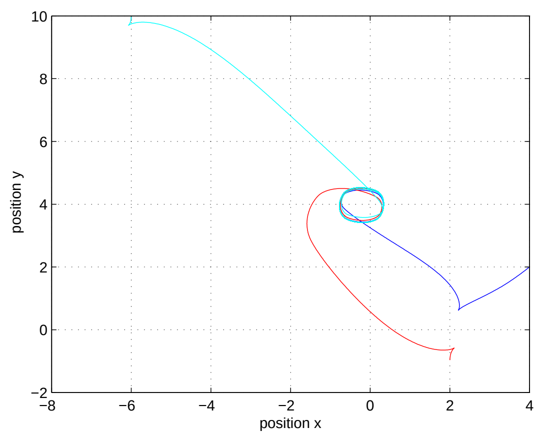

Fig. 2. The 3 unicycles rendezvous under the matrix control C_1 (directed graph $\mathcal{G}$)

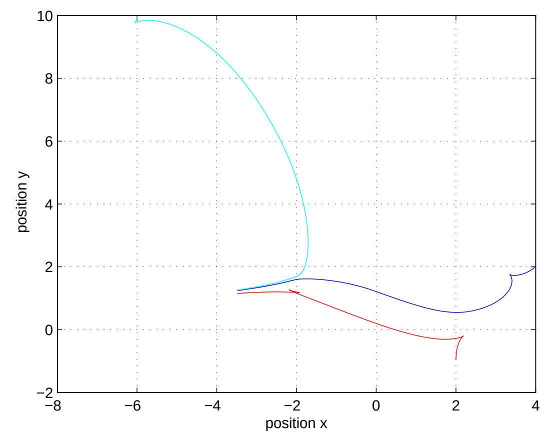

Fig. 3. The 3 unicycles rendezvous under the matrix control C_2 (directed graph $\mathcal{G}$)

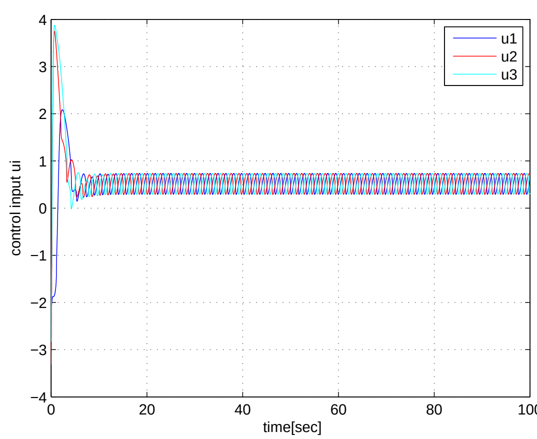

Fig. 4. Protocols u_i as designed in (33) under C_1 (directed graph $\mathcal{G}$)

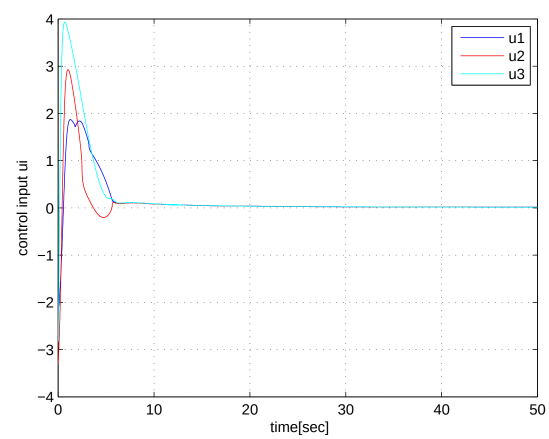

Fig. 5. Protocols u_i as designed in (33) under C_2 (directed graph $\mathcal{G}$)

The simulation result sketched in Fig. 2 is obtained under the protocol (33,34) with the control matrix C_1, while Fig. 3 shows the results under the protocol (33,35) with C_2. The matrix C must be designed following to Assumption II.7, but as we see in figures Fig. 2-3, it plays an important

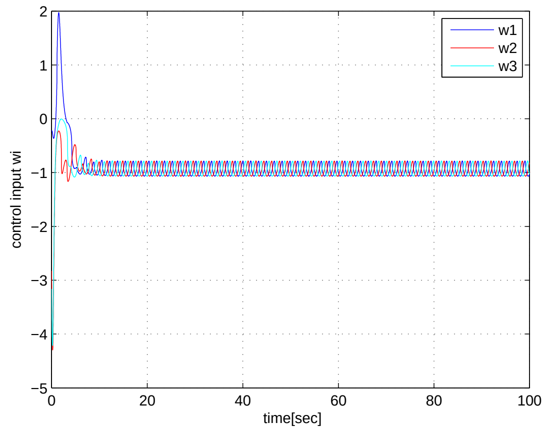

Fig. 6. Protocols w_i as designed in (34) under C_1 (directed graph $\mathcal{G}$)

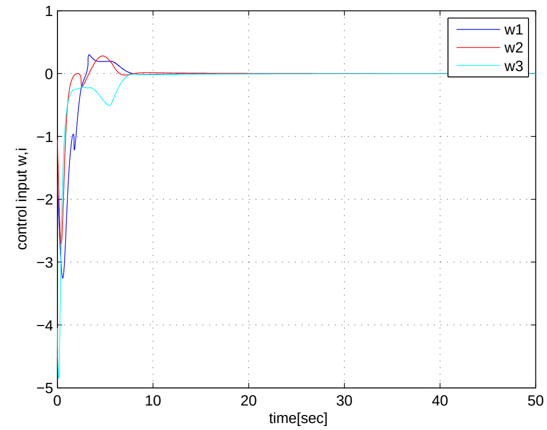

Fig. 7. Protocols w_i as designed in (35) under C_2 (directed graph $\mathcal{G}$)

role in the behaviors of agents, at least for the rendezvous problem. This can be viewed in figures Fig. 4-5 for the protocols u_i and figures Fig. 6-7 with the protocols w_i, obtained for C_1 and C_2, respectively. The decision can be toward the C_2 control matrix as there is no oscillation after the establishment time.

Fixed undirected graph (Fig. 1b). The four unicycles are initialized as,

$$(x_1, y_1, \theta_1)(t = 0) = (4, 2, \frac{\pi}{4})$$
$$(x_2, y_2, \theta_2)(t = 0) = (2, -1, -\frac{\pi}{2})$$
$$(x_3, y_3, \theta_3)(t = 0) = (1, 8, \frac{2\pi}{3})$$
$$(x_4, y_4, \theta_4)(t = 0) = (-1, -4, \pi)$$

The case of four unicycles obey to an undirected graph presented in Fig. 8 (using C_1) and Fig. 9 (with C_2). Clearly the control matrix C of the protocol affects the behavior of the consensus and ensures the rendezvous. Figures Fig. 10-11 show the protocols u_i under C_1 and C_2, respectively. Likewise, figures Fig. 12-13 sketch the protocol w_i. The results are obtained with an undirected topology. In terms of the protocol amplitudes and times of establishment, improvement is obtained with the C_2 matrix. However, comparison can be made between Fig.

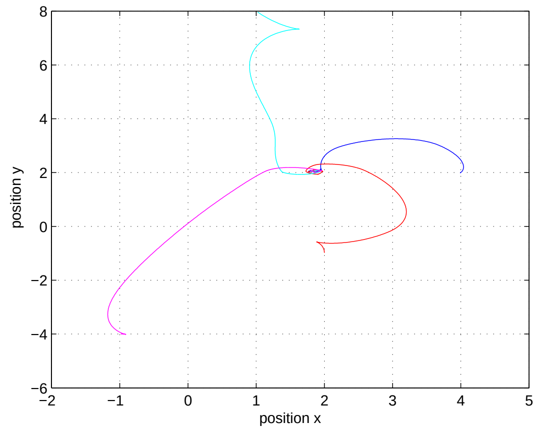

Fig. 8. The 4 unicycles rendezvous (undirected graph $\mathcal{G}$) under the matrix control C_1

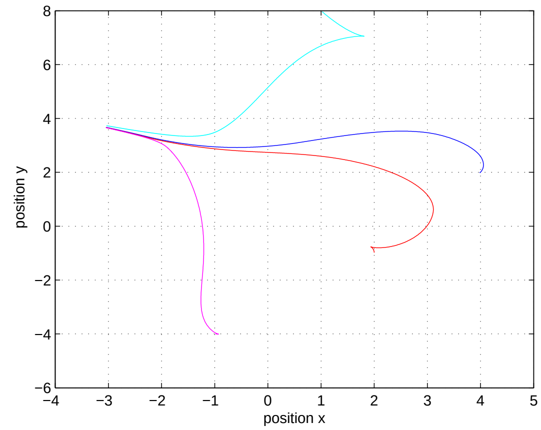

Fig. 9. The 4 unicycles rendezvous (undirected graph $\mathcal{G}$) under the matrix control C_2

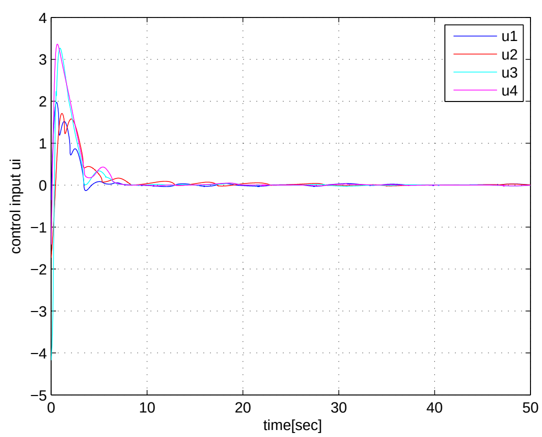

Fig. 10. Protocols u_i as designed in (33) under C_1 (undirected graph $\mathcal{G}$)

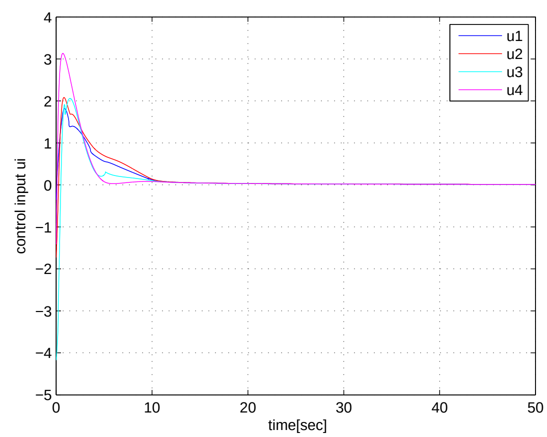

Fig. 11. Protocols u_i as designed in (33) under C_2 (undirected graph $\mathcal{G}$)

5 and Fig. 11 where the establishment time is reduced by half using the directed graph instead of the undirected one.

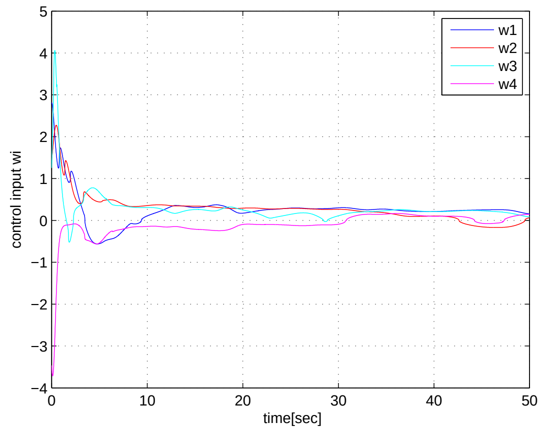

Fig. 12. Protocols w_i as designed in (34) under C_1 (undirected graph $\mathcal{G}$)

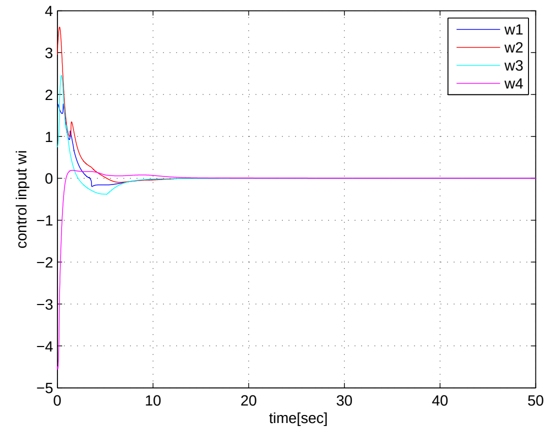

Fig. 13. Protocols w_i as designed in (35) under C_2 (undirected graph $\mathcal{G}$)

B. Multi-agent with second-order dynamic

Consider a second-order dynamic of an agent $(i \in \mathcal{I})$

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= u_i \end{aligned} \tag{36}$$

where $x_i \in \mathbb{R}^n$ denotes the position, $v_i \in \mathbb{R}^n$ it is time derivative, and $u_i \in \mathbb{R}^n$ is the control input. The dynamics (36) take the form given by (2), and it will be treated as multi-Σ_2, with $\mathbf{x}_i = \begin{pmatrix} x_i \\ v_i \end{pmatrix}$, $f_i(\mathbf{x}_i) \begin{pmatrix} v_i \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Condition (5) on f_i can be easily verified (Assumption II.8).
Taking $C = (1 \quad 1)$, from protocol (4) and Proposition IV.2, we are able to propose the following,

$$u_i = -\varphi_\alpha(\sum_{j=1}^{N} a_{ij}(x_i - x_j)) - \varphi_\alpha(\sum_{j=1}^{N} a_{ij}(v_i - v_j)) \tag{37}$$

The double integrator (36) under u_i achieves consensus in positions and velocities. Note that the finite-time consensus for multi-agent networks with second-order agent dynamics as given by (36) was studied by Wang *et al.* [16] in the case of undirected graph. The consensus protocol proposed here for the double integrator is a direct application of Proposition IV.2, and is different from that given in [16].

Fixed directed graph (Fig. 1a). Numerical simulation is presented to illustrate consensus of three agents through the graph (Fig. 1a). The control parameter $\alpha = 0.5$, and each agent initial position vector is as

$$(x_1, x_2, x_3)(t = 0) = (5, 10, 1)(m)$$

and the initial velocity vector is

$$(v_1, v_2, v_3)(t = 0) = (2, -1, 8)(m/s)$$

Consider a fixed directed graph (Fig. 1a), figures in Fig.14 and Fig. 15 show the effectiveness of the given protocol (37) as positions and velocities achieve consensus in finite time.

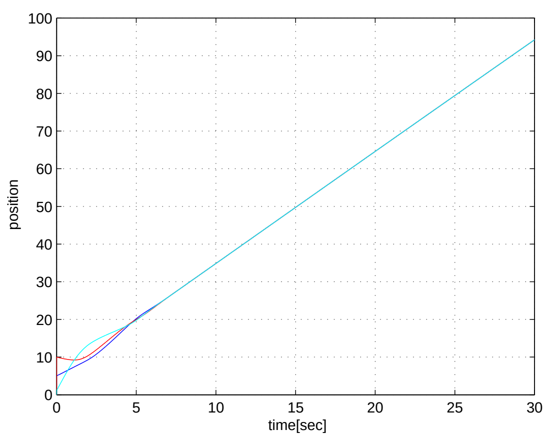

Fig. 14. Consensus in positions of 3 second order dynamics (directed graph).

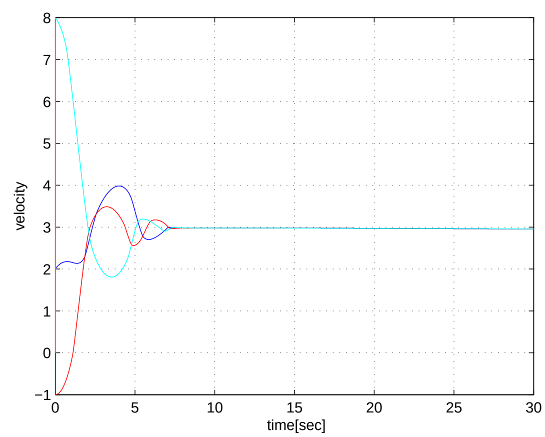

Fig. 15. Consensus in velocities of 3 second order dynamics (directed graph).

Fixed undirected graph (Fig. 1b). Numerical simulations are presented with the control parameter $\alpha = 0.5$, the initial position vector is

$$(x_1, x_2, x_3, x_4)(t = 0) = (5, 10, 1, -5)(m)$$

and the initial velocity vector is

$$(v_1, v_2, v_3, v_4)(t = 0) = (2, -1, 8, -4)(m/s)$$

The 4 double integrators (36) with the protocol (37) meet finite-time consensus in positions and velocities, as given by figures Fig. 16-17. The establishment time for the directed graph (Fig.

16-17) is less important than the undirected graph (Fig. 14-15). The amplitudes of velocities (Fig. 17) are more significant for the undirected graph in comparison to the directed one, as given by Fig. 15. To solve the finite-time consensus problem, protocols given by figures Fig. 18-19 imply the subsequent observations which promote the directed graph in terms of times and amplitudes in multi-second order dynamics.

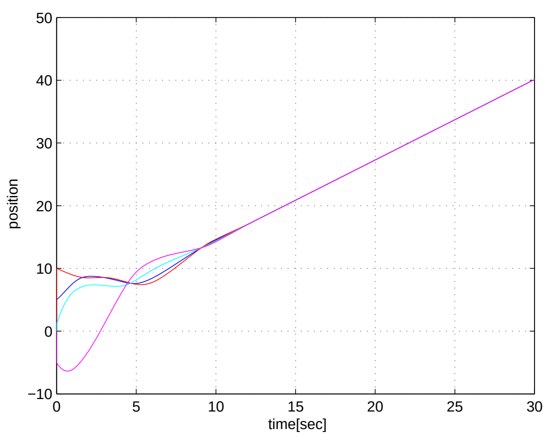

Fig. 16. Consensus in positions for 4 agents as second order dynamics (undirected graph).

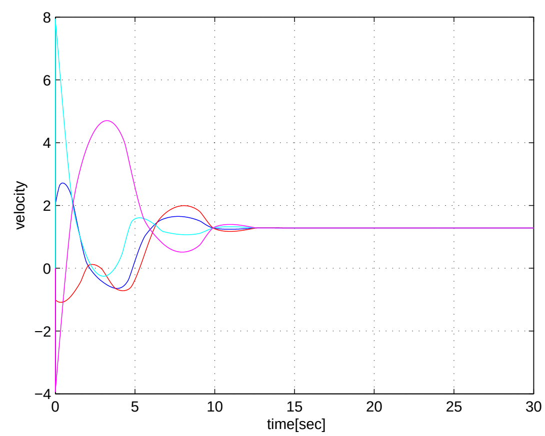

Fig. 17. Consensus velocities for 4 agents as second order dynamics (undirected graph).

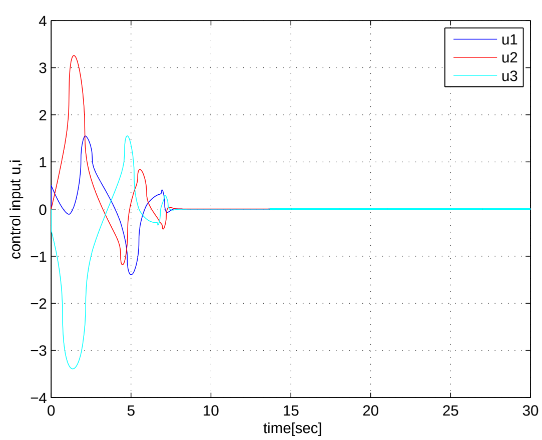

Fig. 18. Protocols u_i as specified by (37), case of the directed graph.

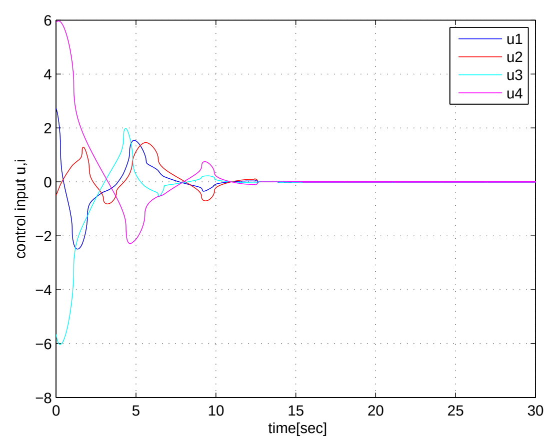

Fig. 19. Protocols u_i as specified by (37), case of the undirected graph.

VI. CONCLUSION

In this paper, controlled dynamic systems in network are described by two main nonlinear and continuous first-order differential equations with/without drift terms. Finite-time consensus are achieved despite complexity in the networked models, and this due to heterogeneity of the components and the vector size of states. Some protocols are proposed and sufficient conditions are established leading to finite-time consensus of controlled nonlinear systems in network. Following to consensus objectives, the multi-system behaviors in simulations, as was given for the multi-unicycle and the multi-second order dynamics, a directed graph presents some advantages such as less amplitude of protocols and low time of establishment. Results for finite-time consensus for homogenous/heterogenous multi-system are also achieved.

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