Finite-time average consensus in networked dynamic systems

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Abstract:
In networked dynamic systems (agents), a consensus is a group of agents try to reach agreement that depends on their states under some interaction rules (protocols) as inputs. When the objective is to agree upon to average, it is an average consensus. The paper addresses the problem of finite-time average consensus in networked dynamic systems. Protocols are presented for high dimensional multi-system, considered as controlled first-order differential equation. The agreement achievement is analyzed through an undirect fixed graph, like an interaction topology. The given examples in simulation show the effectiveness of the proposed protocols.

1. INTRODUCTION

In the past few years, the cooperative control problem for a group of agents is a popular research topic in decentralized control. Many applications can be found in various area: rendezvous problem of multi-vehicle, control of training, flocking, attitude the synchronization, the fusion of sensors. Is one of the main challenges in cooperative design decentralized control systems, such as some objectives of the group can be achieved. The coherent movement in masses is called consensus. Thus, the problem of consensus plays a central role in study of multi-agent systems. Early works on the consensus problem of multi-agent systems can be found in Vicsek [1995], Jadabaie [2003], Saber [2004], Ren [2005] and Xiao [2006].
A special case of the consensus problem in multi-agent systems is the finite-time consensus problem, which is sufficiently studied in the literature (Cortes [2006],Hui [2008],Wang [2010],Xiao [2009],Wang [2008],Yougcan [2011]. Nevertheless, the finite-time consensus problem that has been solved so far is mostly only for agents with first or second order dynamics, in (Zoghlami [2013] et al. and Zoghlami [2014] et al.) the authors treated finite-time consensus for nonlinear networked systems where each systems is modeled by drift/driftless systems.
An interesting topic in consensus problem is the average consensus problem means to design a networked interaction protocol such that the states of all the agents converge (asymptotically/ finite time) to the average of their states (Zhu [2010], Fangcui [2011],Shahram [2012] and Shuai [2013]), to name just a few.

In this paper, we investigate the finite-time average consensus problem. Using an undirected fixed graph, the average consensus study for a nonlinear networked systems remains a challenge problem. Further, the research is motivated by the fact that each dynamic system is taken highly nonlinear with/without drift term in the model. Inspired from finite-time stability results presented in Bhat [2000], Hong [2006] and the graph theory Cremean [2003], nonlinear consensus protocols are proposed throughout the paper.

The paper is organized as follows. Some preliminaries results, the problem statement, and the finite-time average consensus protocol are formulated in section 2. In section 3 one solves a finite-time average consensus of multi-system without drift terms. The finite-time average consensus of multi-system with drift is detailed in section 4. Finally, illustrative examples are presented in section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

Throughout the paper, we use $\mathbf{R}$ to denote the set of real number. $\mathbf{R}^n$ is the n-dimensional real vector space and $\|.\|$ denotes the Euclidian norm. $\mathbf{R}^{n\times n}$ is the set of $n\times n$ matrices. $diag\{m_1, m_2, ..., m_n\}$ denotes a $n\times n$ diagonal matrix. $I_n \in \mathbf{R}^{n\times n}$ is the identity matrix. The symbol $\otimes$ is the Kronecker product of matrices. We use $sgn(.)$ to denote the signum function. For a scalar x, note that $\varphi_\alpha(x) = sgn(x)|x|^\alpha$. We use $\mathbf{x} = (x_1, ..., x_n)^T$ to denote the vector in $\mathbf{R}^n$. Let $\phi_\alpha(\mathbf{x}) = (\varphi_\alpha(x_1), ..., \varphi_\alpha(x_n))^T$, and $\mathbf{1}_n = (1, ..., 1)^T$. The exponent T is the transpose.

2.1 Graph theory

In this subsection, we introduce some basic concepts in algebraic graph theory for multi-agent networks. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a directed graph, where $\mathcal{V} = \{1, 2, ..., n\}$ is the

set of nodes, node i represents the ith agent, $\mathcal{E}$ is the set of edges, and an edge in $\mathcal{G}$ is denoted by an ordered pair (i, j). $(i, j) \in \mathcal{E}$ if and only if the ith agent can send information to the jth agent directly.
$A = [a_{ij}] \in \mathbf{R}^{n\times n}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements, where $a_{ij} > 0$ if there is an edge between the ith agent and jth agent and $a_{ij} = 0$ otherwise. Moreover, if $A^T = A$, then $\mathcal{G}$ is also called an undirected graph. In this paper, we will refer to graphs whose weights take values in the set$\{0, 1\}$ as binary and those graphs whose adjacency matrices are symmetric as symmetric. Let $D = diag\{d_1, ..., d_n\} \in \mathbf{R}^{n\times n}$ be a diagonal matrix, where $d_i = \sum_{j=1}^{n} a_{ij}$ for $i = 0, 1, ..., n$. Hence, we define the Laplacian of the weighted graph

$$L = D - A \in \mathbf{R}^{n\times n}$$

The undirected graph is called connected if there is a path between any two vertices of the graph. Note that time varying network topologies are not considered in this paper.

2.2 Some useful lemmas

In order to establish our main results, we need to recall the following Lemmas.

Lemma 1. Bhat [2000]. Consider the system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f(0) = 0$, $\mathbf{x} \in \mathbf{R}^n$, there exist a positive definite continuous function $V(\mathbf{x}) : U \subset \mathbf{R}^n \to \mathbf{R}$, real numbers $c > 0$ and $\alpha \in]0, 1[$, and an open neighborhood $U_0 \subset U$ of the origin such that $\dot{V} + c(V(\mathbf{x}))^\alpha \leq 0$, $\mathbf{x} \in U_0\backslash\{0\}$. Then $V(\mathbf{x})$ converges to zero in finite time. In addition, the finite settling time T satisfies $T \leq \dfrac{V(\mathbf{x}(0))^{1-\alpha}}{c(1-\alpha)}$.

Lemma 2. Saber [2004]. For a connected undirected graph $\mathcal{G}$, the Laplacian matrix L of $\mathcal{G}$ has the following properties, $\mathbf{x}^T L\mathbf{x} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x_i - x_j)^2$, which implies that L is positive semi-definite. 0 is a simple eigenvalue of L and $\mathbf{1}$ is the associated eigenvector. Assume that the eigenvalues of L are denoted by $0, \lambda_2, ..., \lambda_n$ satisfying $0 \leq \lambda_2 \leq ... \leq \lambda_n$. Then the second smallest eigenvalue satisfies $\lambda_2 > 0$. Furthermore, if $\mathbf{1}^T\mathbf{x} = 0$, then $\mathbf{x}^T L\mathbf{x} \geq \lambda_2 \mathbf{x}^T\mathbf{x}$.

Lemma 3. Hardy [1952]. Let $x_1, x_2, ..., x_n \geq 0$ and $0 < p \leq 1$. Then $(\sum_{i=1}^{n} x_i)^p \leq \sum_{i=1}^{n} x_i^p \leq n^{1-p}(\sum_{i=1}^{n} x_i)^p$.

2.3 Problem statements

We study the finite-time average consensus of two-types of networked systems. The first type is given by equation (1) which describes a controlled system without drift. The second type is represented by equation (2) which is clearly a controlled linear system with drift. One notes that the matrix B for the two models depends on the system's states.

Consider a group of N high-dimensional agents where each agent's behavior is described by a controlled nonlinear model without drift Σ_1, considered as given by dynamic (1) and system Σ_2 with drift as shown by dynamic (2), $\forall i \in \mathcal{I} = \{1, ..., N\}$

$$\Sigma_1 : \quad \dot{x}^i = B(x^i)u^i \tag{1}$$

and

$$\Sigma_2 : \quad \dot{x}^i = f^i(x^i) + B(x^i)u^i \tag{2}$$

where $x^i \in \mathbf{R}^n$, and for $1 \leq i \leq N$ $x^i = [x_1^i, x_2^i, ..., x_n^i]^T$, $B(x^i) \in \mathbf{R}^{n\times m}$, the continuous maps $f^i : \mathbf{R}^n \to \mathbf{R}^n$, $u^i \in \mathbf{R}^m$ is the input, and $B(x^i) = [b_{kl}]$ for $1 \leq k \leq n$ and $1 \leq l \leq m$.

Definition 4. Given a control-input u^i, we say that systems in networks meet a finite-time average consensus if for any system's state initial conditions, there exists some finite time T such that:

$$\lim_{t\to T} \|x^i(t) - \chi(t)\| = 0 \tag{3}$$

for any $i \in \mathcal{I}$, and where $\chi(t) = \dfrac{1}{N}\sum_{j=1}^{N} x^j(t)$.

$\chi(t)$ can be interpreted as the instantaneous consent providing and serves the group objective.
For multi-Σ_1 and multi-Σ_2 systems, one might analyze the following consensus protocol candidate.

For $i \in \mathcal{I}$, the consensus protocol candidate is given by,

$$u^i = -C(x^i)\sum_{j=1}^{N} a_{ij}\phi_\alpha(x^i - x^j) \tag{4}$$

where the a_{ij} elements are of the $\mathcal{G}$ adjacency matrix, $\alpha \in]0, 1[$, and $\phi_\alpha(.)$ is defined in section 2. $C(x^i) \in \mathbf{R}^{m\times n}$ is such that the following assumption hold..

Assumption 5. $C(x^i)$ is such that the matrix product $B(x^i)C(x^i)$ is positive semi-definite and diagonalizable matrix.

Throughout the paper, one denotes by $\tilde{B} = B(x^i)C(x^i)$ where $\tilde{B} = [\tilde{b}_{mk}]_{m,k}$ for $1 \leq m, k \leq n$.

3. THE MULTI-Σ_1 FINITE-TIME AVERAGE CONSENSUS

For finite-time average consensus of multi-Σ_1 one considers, as interaction topology an undirected fixed graph, each Σ_1 vector state to compute the average vector, and the protocol candidate (4). As the matrix B structure is taken identical for each Σ_1, than one might think to networked homogeneous systems.

Theorem 6. Let $\mathcal{G}$ be an undirected and connected graph, under the protocol (4) and Assumption 5 the multi-Σ_1 achieves a finite-time average consensus in the sense of (3).

Proof. We introduce $\xi^i(t) = x^i(t) - \chi(t)$. Let us first calculate the time derivative of $\chi(t)$,

$$\begin{aligned}\dot{\chi}(t) &= \frac{1}{N}\sum_{i=1}^{N}\dot{x}^i(t)\\ &= -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\tilde{B}\phi_\alpha(x^i - x^j)\\ &= -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\tilde{B}\phi_\alpha(x^i - x^j)\\ &\quad -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\tilde{B}\phi_\alpha(x^i - x^j)\end{aligned} \tag{5}$$

As $a_{ij} = a_{ji}$ (undirected graph) and ϕ_α is an odd function, then it is straightforward to verify that the last equality in (5) leads to $\dot{\chi}(t) = 0$. Therefore, $\dot{\xi}^i(t) = \dot{x}^i(t)$.
Taking the Lyapunov function, let $\xi(t) = (\xi^1, ..., \xi^N)$

$$V(\xi(t)) = \frac{1}{2}\xi^T\xi = \frac{1}{2}\sum_{i=1}^{N}(\xi^i)^T\xi^i \tag{6}$$

Due to the fact that $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq N$, we have

$$\begin{aligned}\dot{V}(\xi(t)) &= \sum_{i=1}^{N}\xi^{iT}\dot{\xi}^i\\ &= -\sum_{i,j=1}^{N}a_{ij}\xi^{iT}\tilde{B}\phi_\alpha(\xi^i - \xi^j)\\ &= -\frac{1}{2}\sum_{i,j=1}^{N}a_{ij}(\xi^i - \xi^j)^T\tilde{B}\phi_\alpha(\xi^i - \xi^j)\end{aligned} \tag{7}$$

Let $\tilde{B} = PDP^{-1}$, where $D = diag\{0, \mu_2(x^i), ..., \mu_n(x^i)\} \in \mathbf{R}^{n\times n}$, and $\mu_2(x^i), ..., \mu_n(x^i)$ are the eigenvalues of the matrix $\tilde{B}$ given in increasing order and such that $\mu_2(x^i) > 0$ for all $x^i \in \mathbf{R}^n$.

Therefore,

$$\begin{aligned}\dot{V}(\xi(t)) &\leq -\frac{1}{2}\sum_{i,j=1}^{N}a_{ij}\mu_2(x^i)\|\xi^i - \xi^j\|^{\alpha+1}\\ &\leq -\frac{1}{2}\sum_{i,j=1}^{N}(a_{ij}\mu_2(x^i))^{\frac{2}{\alpha+1}}\|\xi^i - \xi^j\|^2)^{\frac{\alpha+1}{2}}\\ &\leq -\frac{1}{2}(\sum_{i,j=1}^{N}(a_{ij}\mu_2(x^i))^{\frac{2}{\alpha+1}}\|\xi^i - \xi^j\|^2)^{\frac{\alpha+1}{2}}\end{aligned} \tag{8}$$

Now we consider $\Theta = [\theta_{ij}] \in \mathbf{R}^{n\times n}$, where $\theta_{ij} = (a_{ij}\mu_2(x^i))^{\frac{2}{\alpha+1}}$. Then by Lemma 2

$$\sum_{i,j=1}^{N}(a_{ij}\mu_2(x^i))^{\frac{2}{\alpha+1}}\|\xi^i - \xi^j\|^2) = 2\xi^T(L(\Theta)\otimes I_n)\xi$$

So,

$$\frac{\xi^T(L(\Theta)\otimes I_n)\xi}{\|\xi\|^2} \geq \lambda_2(L(\Theta)) > 0 \tag{9}$$

recall that $L(\Theta)$ is the graph Laplacian of the undirected weighted graph $\mathcal{G}(\Theta)$. Therefore, we can rewrite the last inequality (8)

$$\begin{aligned}\dot{V}(\xi(t)) &\leq -2^{\frac{\alpha-1}{2}}(\xi^T(L(\Theta)\otimes I_n)\xi)^{\frac{\alpha+1}{2}}\\ &\leq -2^{\frac{\alpha-1}{2}}(\frac{\xi^T(L(\Theta)\otimes I_n)\xi}{\|\xi\|^2})^{\frac{\alpha+1}{2}}(V(\xi(t)))^{\frac{\alpha+1}{2}}\\ &\leq -2^{\frac{\alpha-1}{2}}\lambda_2^{\frac{2}{\alpha+1}}(V(\xi(t)))^{\frac{\alpha+1}{2}}\end{aligned} \tag{10}$$

Consequently, by Lemma 1, V reaches zero at an estimated finite time

$$T(\xi(0)) = \frac{V(\xi(0))^{\frac{1-\alpha}{2}}}{2^{\frac{\alpha-3}{2}}\lambda_2^{\frac{2}{\alpha+1}}(1-\alpha)}$$

As a result the multi-Σ_1 networked dynamic systems with the protocol (4) solve a finite-time average consensus. This ends the proof.

4. THE MULTI-Σ_2 FINITE-TIME AVERAGE CONSENSUS

The multi-Σ_2 in network is based on dynamic (2) while the consensus protocol candidate is given by (4). Recall that the Σ_2 dynamic as given by (2) is currently present in controlled autonomous systems. Further, the drift term will be considered as linear with respect to the Σ_2's states. Note that f^i in (2) can be different for each Σ_2 dynamic systems. Than one might think to networked heterogeneous systems.

Case1: Linear drift term
we consider equation (2) and let $f^i(x^i) \equiv \tilde{A}x^i$, system (2) becomes in the form

$$\dot{x}^i = \tilde{A}x^i + B(x^i)u^i \tag{11}$$

where $\tilde{A} \in \mathbf{R}^{n\times n}$ with $\tilde{A} = [\tilde{a}_{p,q}]_{1\leq p,q\leq n}$.

Theorem 7. Let $\mathcal{G}$ be an undirected and connected graph. Under the protocol (4) and Assumption 5 the multi-Σ_2 achieves a finite-time average consensus in the sense of (3).

Proof. One introduces $\xi(t) = x^i(t) - \chi(t)$. The goal is to rewrite equation (11) in closed loop depending on ξ^i and to prove that ξ converges to zero in finite time.
Since $a_{ij} = a_{ji}$ and ϕ_α is an odd function, then we have

$$\begin{aligned}\dot{\chi} &= \frac{1}{N}\sum_{i=1}^{N}(\tilde{A}x^i + Bu^i)\\ &= \frac{1}{N}\sum_{i=1}^{N}\tilde{A}x^i + \frac{1}{N}\sum_{i=1}^{N}Bu^i\\ &= \frac{1}{N}\sum_{i=1}^{N}\tilde{A}x^i\end{aligned} \tag{12}$$

Note that $\sum_{i=1}^{N}Bu^i = 0$, and this is due to (4)-(5).
Consequently,

$$\begin{aligned}\dot{\xi}^i &= \tilde{A}\xi^i + Bu^i\\ &= \tilde{A}\xi^i - \tilde{B}\sum_{j=1}^{N}a_{ij}\phi_\alpha(\xi^i - \xi^j)\end{aligned} \tag{13}$$

Using the Lyapunov function (6), and consider the time derivative of $V(\xi)$ along the networked system trajectories (13), we may write

$$\begin{aligned}\dot{V}(\xi(t)) &= \sum_{i=1}^{N} \xi^{iT}\dot{\xi}^i \\ &= \sum_{i=1}^{N} \xi^{iT}\tilde{A}\xi^i - \sum_{i,j=1}^{N} a_{ij}\xi^{iT}\tilde{B}\phi_\alpha(\xi^i - \xi^j) \\ &\leq \|\tilde{A}\|_\infty \sum_{i=1}^{N} \|\xi^i\|^2 - 2^{\frac{\alpha-1}{2}}\lambda_2^{\frac{2}{\alpha+1}}(V(\xi(t)))^{\frac{\alpha+1}{2}} \\ &\leq \|\tilde{A}\|_\infty V(\xi(t)) - 2^{\frac{\alpha-1}{2}}\lambda_2^{\frac{2}{\alpha+1}}(V(\xi(t)))^{\frac{\alpha+1}{2}} \\ &\leq -V(\xi(t))^{\frac{\alpha+1}{2}}[2^{\frac{\alpha-1}{2}}\lambda_2^{\frac{2}{\alpha+1}} - \|\tilde{A}\|_\infty(V(\xi(t)))^{\frac{1-\alpha}{2}}]\end{aligned} \tag{14}$$

where $\|\tilde{A}\|_\infty = \max_{1\leq p\leq n}\sum_{q=1}^{n}|\tilde{a}_{pq}| > 0$. Since $\frac{1-\alpha}{2} > 0$ and V is continuous function which takes 0 at the origin ($\xi \equiv 0$), there exists an open neighborhood Ω of the origin and the last inequality (14) yields to

$$\dot{V}(\xi(t)) \leq -2^{\frac{\alpha-3}{2}}\lambda_2^{\frac{2}{\alpha+1}}V(\xi(t))^{\frac{\alpha+1}{2}} \tag{15}$$

by Lemma 2, V reaches zero at an estimated finite time

$$T(\xi(0)) = \frac{V(\xi(0))^{\frac{1-\alpha}{2}}}{2^{\frac{\alpha-5}{2}}\lambda_2^{\frac{2}{\alpha+1}}(1-\alpha)}$$

Therefore the networked system based on model (11) and the protocol (4) leads to a finite-time consensus. This ends the proof.

Case2: Nonlinear drift term
In this case, we consider the system (2) and the drift term is nonlinear and we assume that f^i is a convex function.

Theorem 8. Let $\mathcal{G}$ be an undirected graph. With the protocol (4) a networked system based on (2) meet a finite-time average consensus.

Proof. As f^i is assumed to be convex, we have

$$f^i(x^i) - \frac{1}{N}\sum_{i=1}^{N} f^i(x^i) \leq f^i(x^i) - f^i(\frac{1}{N}\sum_{i=1}^{N} x^i)$$

Moreover f^i is locally lipschitz function in an open set $\Omega \subset \mathbf{R}^n$ containing ξ. Therefore,

$$\begin{aligned}\|f^i(x^i) - \frac{1}{N}\sum_{i=1}^{N} f^i(x^i)\| &\leq \|f^i(x^i) - f^i(\chi)\| \\ &\leq K_1\|\xi^i\|\end{aligned} \tag{16}$$

where $K_1 > 0$ is the lipschitz's constant. Now, for convenience the Lyapunov function is given by (6) and the following inequality is obtained

$$\dot{V}(\xi(t)) \leq -2^{\frac{\alpha-3}{2}}\lambda_2^{\frac{2}{\alpha+1}}V(\xi(t))^{\frac{\alpha+1}{2}} \tag{17}$$

At this stage, one concludes that the multi-Σ_2 issues from (2) with the protocol (4) leads to a finite-time average consensus. This ends the proof.

5. ILLUSTRATIVE EXAMPLES

Two illustrative examples are considered where the multi-unicycle represents networked system modeled by (1) (driftless) and the multi-system based on second order dynamic which imply networked multi-model of type (2) (with drift). Each associated protocol is deduced from (4) and results for the finite-time average consensus case given by the protocol (4) are shown under the proposed undirected graph Fig.1,

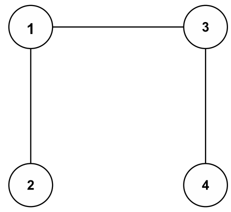


Fig. 1. $\mathcal{G}$ for a system with 4 agents.

5.1 Finite-time average consensus for multi-unicycle

Consider four wheeled mobile robots (unicycles) where the i^{th} nonholonomic kinematic model is as:

$$\begin{pmatrix}\dot{x}_i\\ \dot{y}_i\\ \dot{\theta}_i\end{pmatrix} = \begin{pmatrix}\cos\theta_i & 0\\ \sin\theta_i & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_i\\ w_i\end{pmatrix} \quad i = 1, ..., 4 \tag{18}$$

where (x_i, y_i, θ_i) denotes the position and the orientation in a an inertial frame. The inputs u_i and w_i are the linear and angular velocities, respectively. Let $B = \begin{pmatrix}\cos\theta_i & 0\\ \sin\theta_i & 0\\ 0 & 1\end{pmatrix}$ and $C = B^T$. Based on Theorem 6, the finite-time average consensus problem can be achieved through the following protocol

$$\begin{aligned}u_i &= -\sum_{j=1}^{4} a_{ij}\varphi_\alpha(x_i - x_j)\cos\theta_i \\ &\quad -\sum_{j=1}^{4} a_{ij}\varphi_\alpha(y_i - y_j)\sin\theta_i \\ w_i &= -\sum_{j=1}^{4} a_{ij}\varphi_\alpha(\theta_i - \theta_j)\end{aligned} \tag{19}$$

where φ_α defined in section 2 and a_{ij} are associated to the above graph Fig.1. These simulation results introduce the following initial conditions $(x_1, y_1, \theta_1)(t = 0) = (4, 2, \frac{\pi}{4})$, $(x_2, y_2, \theta_2)(t = 0) = (2, -1, -\frac{\pi}{2})$, $(x_3, y_3, \theta_3)(t = 0) = (1, 8, \frac{2\pi}{3})$. $(x_4, y_4, \theta_4)(t = 0) = (-1, -4, \pi)$.

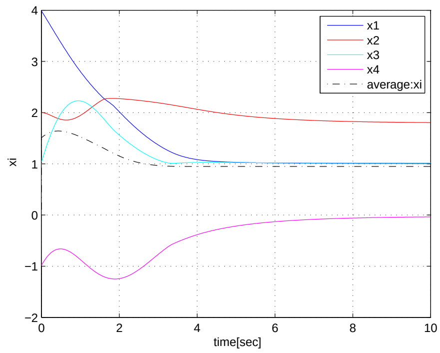
5.2 Finite-time average consensus for multi-second-order dynamics

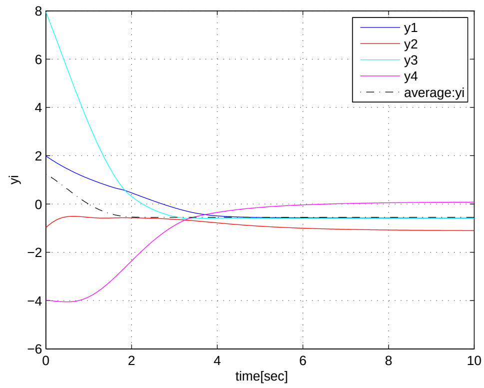
Consider a second-order agent dynamic

$$\begin{aligned}\dot{x}_i &= v_i \\ \dot{v}_i &= u_i \quad i = 1, ..., 4\end{aligned} \tag{20}$$

where $x_i \in \mathbf{R}$ denotes the position, $v_i \in \mathbf{R}$, and $u_i \in \mathbf{R}$ is the control input. The dynamic (20) takes the form given by (11) with:

$\mathbf{x}_i = \begin{pmatrix}x_i\\ v_i\end{pmatrix}$, $f^i(\mathbf{x}_i)\begin{pmatrix}v_i\\ 0\end{pmatrix}$ and $B = \begin{pmatrix}0\\ 1\end{pmatrix}$.





For the protocol (4), $C = (1\ 1)$ and the results of Theorems 7-8, we are able to propose the following:

$$u_i = -\sum_{j=1}^{N} a_{ij}(\varphi_\alpha(x_i - x_j) - \varphi_\alpha(v_i - v_j)) \qquad (21)$$

For a fixed undirected graph (Fig.1), the double integrator (20) under u_i achieves average consensus in positions and velocities.The consensus protocol proposed here for the double integrator is a direct application of Theorem 7.

Four agents through the graph (Fig. 1), the control parameter is taken $\alpha = 0.5$, and agents initial positions are $(x_1, x_2, x_3, x_4)(t = 0) = (5, 10, 1, -5)$ (meter) and initial velocities are $(v_1, v_2, v_3, v_4)(t = 0) = (2, -1, 8, -4)$ (meter/second).
As shown by figures in Fig.3 positions and velocities consent the average.

Remark 9. Other processes can be studied, and where the average is an agreement value of states like a common temperature of sensors where fluctuations of data is important. The energy consumption is also an important factor for stability of electric generators in networks. As example, for a multi-second-order dynamics, the kinetic energies consent an average, and this is shown by figure Fig.4

6. CONCLUSION

In this work, a controlled dynamic model of networked autonomous systems is formulated by two-types of nonlinear and continuous first-order differential equations. Some protocols are proposed and sufficient conditions are achieved covering high dimensional networked homogeneous dynamics. The results lead to a finite-time average consensus of the group. It is interesting to see the case of networked heterogeneous multi-system that treats multi-Σ_1-Σ_2.

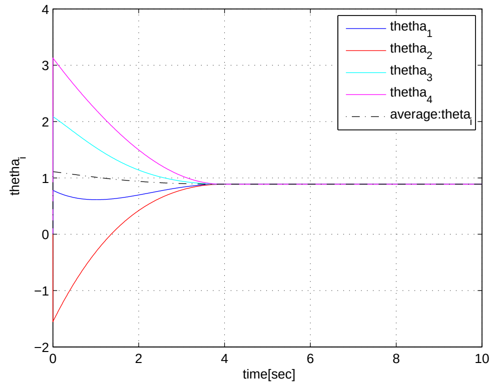


Fig. 2. Finite-time average Consensus results for 4 unicycles

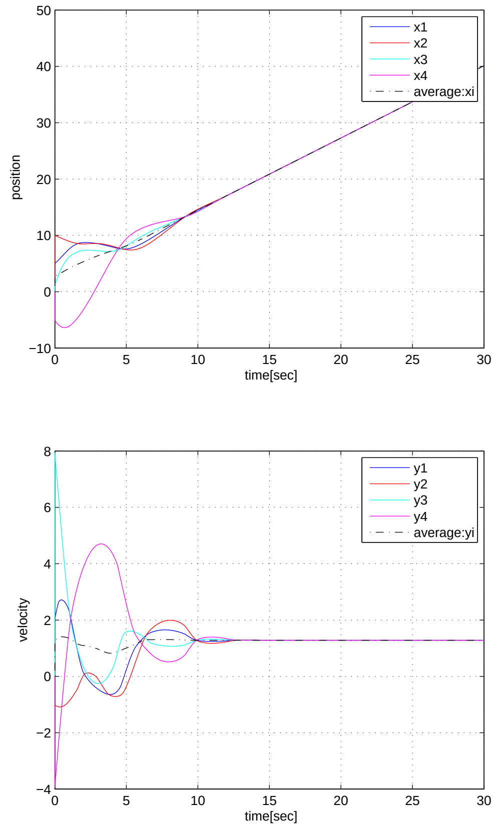


Fig. 3. The average consensus for 4 second-order dynamics

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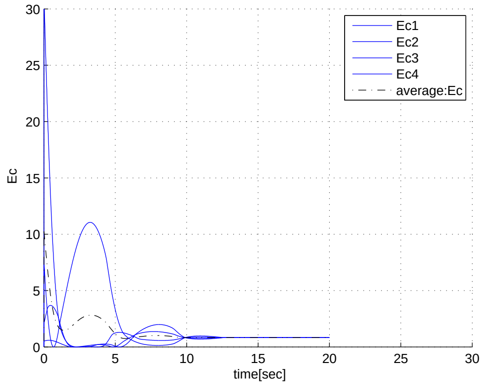

Fig. 4. The average consensus for 4 second-order dynamics

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