MOTION PLANNING OF A FULLY ACTUATED UNMANNED AERIAL VEHICLE

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Abstract : The objective of this paper is to generate a desired flight path to be followed by an Unmanned Aerial Vehicle (UAV), with specified boundary conditions. The space is supposed without obstacles. After the user has defined the goal tasks, the path generator then determines a path for the vehicle that is a trajectory in space. The problem of path planning is formulated as an optimization problem: minimum energy, minimum acceleration and minimum jerk curves.

INTRODUCTION

Unmanned aerial vehicles are a new focus of research because of their important application potential. They can be divided into three different types : reduced scale fixed wing vehicles (airplanes), rotary wing aircraft (helicopter) or lighter than air (airships).
A basic problem which has to be solved by autonomous vehicles is the problem of motion planning. Motion planning means the generation and execution of a plan for moving from one location to another location in space to accomplish a desired task. Moreover, it is desirable that the plan makes optimal use of the available resources to achieve the goal optimizing some 'cost' measure : the time required for the execution of the trajectory, its length, the deviation from a reference trajectory, control effort or energy ... In this paper, trajectories that minimize an appropriate measure of smoothness in the form of an integral cost function are proposed. Depending on the chosen integrand, boundary conditions on the derivatives of the desired order can be enforced. The motion generation module generates a nominal state space trajectory and a nominal control input.
When planning Cartesian trajectories, it is usually possible to characterize the performance of different trajectories[1,2,3,5,6,9]. It is natural to regard the trajectory planning as a variational problem, where the goal is to find a trajectory between given starting and ending positions and orientations that minimizes a chosen cost function. Additional boundary conditions may be specified.
The set of all 3D rigid body displacements forms a Lie group. This group is generally referred to as SE(3), the special Euclidean group in 3D. The tangent space at the identity endowed with the Lie bracket operation has the structure of a Lie algebra and is denoted by se(3).
Curves that minimize the energy between two given points are of particular interest. Such curves are called geodesics and can be considered as a generalization of straight lines in Euclidean space $\Re^n$ to Riemannian manifolds. Some of the geodesics for the scale dependent left invariant metric are screw motions. Since Chasles theorem guarantees the existence of a screw motion between any two points on SE(3), a natural question is whether there exists a metric for which every geodesic is a screw motion. The main result of[3,10] is that there are no Riemannian metric with such a property. It contains a discussion of Riemannian metrics in SE(3). In the context of kinematics, the motion of a rigid body is a curve on SE(3), and the velocity at any point is the tangent vector to the curve at that point.
This article is concerned with methods of computing a trajectory in 6 degrees of freedom space that describes the desired motion. We present two different methods: the first one is based on Singular Value Decomposition while the second one is based on polynomial interpolation by minimizing the traveling time, given realistic constraints, the generated torques/forces and velocities.

KINEMATICS

Consider a rigid body moving in free space. Assume any inertial reference frame {F} fixed in space and a frame {M} fixed to the body at a point O'. At each instant, the configuration (position and orientation) of the rigid body can be described by a homogeneous transformation matrix corresponding to the displacement from frame {F}to frame {M}. The set of all such matrices is called SE(3), the special Euclidean group of rigid body transformations in 3D.

$$SE(3) = \left\{ A \middle| A = \begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix},\ R \in \Re^{3*3},\ d \in \Re^3,\ R^T R = I,\ \det(R) = 1 \right\} \quad \textbf{eq 1}$$

SE(3) is a Lie group for the standard matrix multiplication and it is a manifold. On any Lie group, the tangent space at the group identity has the structure of a Lie algebra. The Lie algebra of SE(3) denoted by se(3), is given by :

$$se(3) = \left\{ \begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix}, \hat{\omega} \in \Re^{3*3},\ v \in \Re^3, \hat{\omega}^T = -\hat{\omega} \right\} \quad \textbf{eq 2}$$

A 3*3 skew symmetric matrix $\hat{\omega}$ can be identified with a vector $\omega \in \Re^3$ so that for any arbitrary vector $x \in \Re^3$, $\hat{\omega} x = \omega \times x$ where $\times$ is the vector cross-product operation in $\Re^3$.

$$\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad \textbf{eq 3}$$

Each element $S \in se(3)$ can thus be identified with a vector pair $\{\omega \quad v\}$.

Given a curve $A(t) : [-a \quad a] \to SE(3)$, an element S(t) of the Lie algebra se(3) can be associated to the tangent vector $\dot{A}(t)$ at an arbitrary point t by :

$$S(t) = A^{-1}(t)\dot{A}(t) = \begin{pmatrix} \hat{\omega} & R^T \dot{d} \\ 0 & 0 \end{pmatrix} \quad \textbf{eq 4}$$

where $\hat{\omega}(t) = R^T(t)\dot{R}(t)$ is the corresponding element from SO(3).

A curve on SE(3) physically represents a motion of the rigid body. If $\{\omega(t) \quad V(t)\}$ is the pair corresponding to S(t), then ω physically corresponds to the angular velocity of the rigid body, while V is the linear velocity of the origin O' of the frame {M}.

$$V = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad \omega = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \quad \textbf{eq 5}$$

We choose a parameterization of SE(3) induced by the product structure SO(3)x$\Re^3$. We define a set of coordinates $\sigma_1, \sigma_2, \sigma_3, d_1, d_2, d_3$ for an arbitrary element $A = (R, d) \in SE(3)$ so that d_1, d_2, d_3 are the coordinates of d in $\Re^3$. Exponential coordinates are chosen as local parameterization of SO(3). For $R \in SO(3)$ sufficiently close to the identity (i.e excluding the points Tr (R)= -1 or Tr (A)= 0 or equivalently rotations through angles of π) we define the exponential coordinates

$$R = \exp(\hat{\sigma}) = I + \frac{\sin\|\sigma\|}{\|\sigma\|}\hat{\sigma} + \frac{1-\cos\|\sigma\|}{\|\sigma\|^2}\hat{\sigma}^2 \quad \textbf{eq 6}$$

$\sigma \in \Re^3$ where $\hat{\sigma}$ is the skew symmetric matrix corresponding to $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. σ is the product of the axis and angle of rotation of R. $\|.\|$ is the standard Euclidean norm.

The set of all positions and orientations being not Euclidean, there is no obvious choice of a metric on this set. The norm of the velocity and the distance between two positions and orientations is not defined. The quadratic form $\omega^T G \omega$ can be interpreted as the (rotational) kinetic energy. Consequently, 2G can be thought as the inertia matrix of a rigid body with respect to a certain choice of the body frame $\{M\}$. Therefore, for an arbitrarily shaped body with inertia matrix 2G we can formulate a positive definite metric with matrix :

$$W = \frac{1}{2}Tr(G)I_3 - G \quad \textbf{eq 7}$$

Thus this gives a formula for constructing an ambient metric space that is compatible with the given structure of SO(3).

Another formulation is possible. In dynamic analysis, the kinetic energy of a rigid body is a scalar invariant and therefore it makes sense to define the matrix W according to the inertial properties of the rigid body.

$$W = \begin{pmatrix} J & 0 \\ 0 & M \end{pmatrix} \quad \textbf{eq 8}$$

Matrix J is the inertia tensor of the rigid body and M is its mass matrix . This metric W is also invariant with respect to the choice of the inertial reference frame and the squared norm of a velocity vector at a point equals to the kinetic energy of the rigid body.

MECHANICAL SYSTEM

In this section, analytic expressions for the forces and moments of a system with added mass and inertia such as an airship are introduced. An airship is a lighter than air vehicle using a lifting gas (helium in this particular case)

The dynamic equations (Euler – Poincaré) are given by :

$$M\dot{v} = -\omega x M v - b(.) + f(u)$$
$$J\dot{\omega} = -\omega x J\omega - v * Mv - \beta(.) + \tau(u) \quad \text{eq 9}$$

where M and J are respectively the vehicle's mass and rotational tensors and τ, β and f, b represent respectively the control and non-conservative torques and forces in body axes.

For a system with added masses, the term $v * Mv$ is non zero. And we can propose

$$b(.) = R^T e_3 (mg - B) - diag(D_V).v$$

$$\beta(.) = (R^T e_3 \times \overline{BG})B - diag(D_\omega).\omega \quad \textbf{eq 10}$$

m is the mass of the airship, the propellers and actuators. M includes both the airship's actual mass as well as the virtual mass elements associated with the dynamics of buoyant vehicles. J includes both the airship's actual inertias as well as the virtual inertia elements associated with the dynamics of buoyant vehicles. As the airship displays a very large volume, its added masses and inertias become very significant. Diag(D_V) is the 3*3 aerodynamics forces diagonal matrix. Diag(D_Ω) is the 3*3 aerodynamics moments diagonal matrix.

$e_3 = (0 \quad 0 \quad 1)^T$ a unit vector.

Be_3 : The 3*1 buoyancy force vector. $B = \rho \Delta g$ where Δ is the volume of the envelope, ρ is the difference between the density of the ambient atmosphere ρ_{air} and the density of the helium ρ_{helium} in the envelope, g is the constant gravity acceleration.

$\overline{BG} = (x_b \quad y_b \quad z_b)$ represents the position of the center of buoyancy with respect to the body fixed frame.

As the airship is a slow moving vehicle in the air, we can assume a linear relationship between the speed and the drag.

If a system is fully actuated, it can be steered along any given curve on the configuration manifold, i.e it is controllable. This is not true in general for an underactuated system. However, an underactuated system can be locally controllable if it enjoys the property of nonholonomy. The existence of nonholonomic constraints translates into the fact that the system can be locally steered along a manifold of dimension larger than the number of independent control inputs.

MOTION PLANNING BASED ON SINGULAR VALUE DECOMPOSITION

This section develops a method for generating smooth boundary conditions for a moving rigid body with specified boundary conditions. The problem is well understood in Euclidean spaces, but it is not clear how these techniques can be generalized to curved spaces. The smoothness properties and the optimality of the trajectories need also to be considered.

PROJECTION ON SO(3)

Proposition 1: Let $B \in \Re^{4*4}$ with the following block partition $B = \begin{pmatrix} B_1 & B_2 \\ 0 & 1 \end{pmatrix}$, $B_1 \in \Re^{3*3}$, $B_2 \in \Re^3$ and U, Σ and V be the singular value decomposition of $B_1 W$, then the projection on SE(3) is given by

$$A = \begin{pmatrix} UV^T & B_2 \\ 0 & 1 \end{pmatrix} \in SE(3) \quad \textbf{eq 11}$$

□

The proof of this proposition can be found in[3, 7].

MOTION PLANNING METHOD BASED ON THE PROJECTION METHOD

In this section we consider trajectories between a starting and a final position and orientation that minimize integral cost functions while possibly satisfying boundary conditions. The cost function can be the kinetic energy of the rigid body or some other measure of smoothness involving velocity or its higher derivatives. In particular, we will be interested in curves $A:[a,b] \rightarrow SE(3)$ that minimize integrals of the form

$$J = \int \left\langle h\left(\frac{dA}{dt}\right), h\left(\frac{dA}{dt}\right) \right\rangle dt \quad \textbf{eq 12}$$

where boundary conditions on A(t) and its derivatives may be specified at the end points a and b. The function h returns a vector field. The necessary conditions for the optimal trajectory will be derived using calculus of variations on manifolds[3,7,9,10].

Geodesic

We illustrate this approach with the example in which the cost function is the energy.

Proposition 2: If A(t) is a geodesic for the metric

$$W = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix} \quad \textbf{eq 13}$$

α, β positive scalars acting like scaling factors for angular velocities and linear velocities, the vector pair $\{\omega, v\}$ corresponding to the velocity vector field $V = \frac{dA}{dt}$ must satisfy the equation

$$\frac{d\omega}{dt} = 0 \qquad \frac{dv}{dt} = -\omega x V \quad \textbf{eq 14}$$

This second equation can be simplified to $\ddot{d} = 0$

□

The proof of this proposition can be found in[3].

Proposition 3: Given two positions and orientations $A_i = \begin{pmatrix} R_i & d_i \\ 0 & 1 \end{pmatrix}, A_f = \begin{pmatrix} R_f & d_f \\ 0 & 1 \end{pmatrix}$

The shortest distance path (geodesic) is given by

$$d(t) = (d_{fi} - d_i)t + d_i$$
$$R(t) = R_1 \exp m(\Omega_0 t)$$
$$B(t) = a_1 t + a_0 \quad \textbf{eq 15}$$
$$a_0 = R_i \qquad a_1 = R_f - R_i$$
$$\Omega_0 = \log m(R_i^T R_f)$$

With the boundary conditions

$$B(0) = R_i \qquad B(1) = R_f \quad \textbf{eq 16}$$

□

Straight calculations lead to the proof of this proposition.

The minimal geodesic on SE(3) consists of the union of the respective geodesics on SO(3) and $\Re^3$.

Minimum acceleration curves

Proposition 4 : Let $A(t) = \begin{pmatrix} R(t) & d(t) \\ 0 & 1 \end{pmatrix}$ be a curve between two prescribed points on SE(3) that has prescribed initial and final velocities. If $\{\omega, V\}$ is the vector pair corresponding to $V = \frac{dA}{dt}$, the curve minimizes the cost function only if the following equation holds :

$$\omega^{(3)} + \omega x \ddot{\omega} = 0 \qquad d^{(4)} = 0$$

where $(.)^{(n)}$ denotes the nth derivative of (.) leading to :

$$A(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0 \quad \textbf{eq 17}$$

with

$$a_0 = A(0); \; a_1 = \dot{A}(0);$$
$$a_2 = 3(A(1) - A(0)) - \dot{A}(1) - 2\dot{A}(0); \quad \textbf{eq 18}$$
$$a_3 = -2(A(1) - A(0)) + \dot{A}(1) + \dot{A}(0)$$

With the boundary conditions

$$B(0) = R(0) \qquad \dot{B}(0) = \dot{R}(0)$$
$$B(1) = R(1) \qquad \dot{B}(1) = \dot{R}(1) \quad \textbf{eq 19}$$

□

Proof of Proposition 4 is obtained by using the same method than in Propositions 2 and 3.

Minimum Jerk curves

The minimum jerk curves between two points is obtained by minimizing the integral of the norm of the Cartesian jerk, provided that the appropriate boundary conditions are given. In particular, minimum jerk trajectories are well defined when the initial and final velocities and accelerations are specified.

Proposition 5: The minimum jerk trajectories in the case when the initial and final velocities and accelerations are prescribed to be zero are given by

$$R(t) = R_1 \exp m(\Omega_0 p(t))$$
$$d(t) = p(t)(d_f - d_i) \quad \textbf{eq 20}$$
$$p(t) = 6t^5 - 15t^4 + 10t^3$$

In the general case: $0 \le t \le 1$

$$A(t) = a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \quad \textbf{eq 21}$$

with

$$a_0 = A(0); \; a_1 = \dot{A}(0); \; a_2 = 0.5\ddot{A}(0)$$
$$a_3 = 10(A(1) - A(0))$$
$$-\left(6\dot{A}(0) + 4\dot{A}(1)\right) + \frac{1}{2}\left(-3\ddot{A}(0) + \ddot{A}(1)\right); \quad \textbf{eq22}$$
$$a_4 = -15(A(1) - A(0))$$
$$+\left(8\dot{A}(0) + 7\dot{A}(1)\right) - \frac{1}{2}\left(-3\ddot{A}(0) + 2\ddot{A}(1)\right);$$
$$a_5 = 6(A(1) - A(0))$$
$$-3\left(\dot{A}(0) + \dot{A}(1)\right) + \frac{1}{2}\left(-\ddot{A}(0) + \ddot{A}(1)\right);$$

With the boundary conditions

$$B(0) = R(0) \qquad \dot{B}(0) = \dot{R}(0) \qquad \ddot{B}(0) = \ddot{R}(0) \quad \textbf{eq23}$$
$$B(1) = R(1) \qquad \dot{B}(1) = \dot{R}(1) \qquad \ddot{B}(1) = \ddot{R}(1)$$

□

Singular Values and Vectors

Proposition 6: If the matrix W is chosen as the identity as well as the initial rotation matrix, the singular value matrix has the following form :

$$\Xi = diag(1, s(t), s(t)) \quad \textbf{eq 24}$$

for the geodesic curve

$$s(t) = \left(1 - 2\left(1 - \cos|\sigma_f|\right)t + 2\left(1 - \cos|\sigma_f|\right)t^2\right)^{1/2} \quad \textbf{eq 25}$$

For the minimal acceleration curve

$$s(t)=\begin{pmatrix} 1-6\left(1-\cos\left|\sigma_f\right|\right)t^2+4\left(1-\cos\left|\sigma_f\right|\right)t^3+ \\ 18\left(1-\cos\left|\sigma_f\right|\right)t^4-24\left(1-\cos\left|\sigma_f\right|\right)t^5+ \\ 8\left(1-\cos\left|\sigma_f\right|\right)t^6 \end{pmatrix}^{1/2} \quad \textbf{eq 26}$$

For the minimal jerk curve

$$s(t)=\begin{pmatrix} 1-12\left(1-\cos\left|\sigma_f\right|\right)t^3+30\left(1-\cos\left|\sigma_f\right|\right)t^4+ \\ -20\left(1-\cos\left|\sigma_f\right|\right)t^5+72\left(1-\cos\left|\sigma_f\right|\right)t^6+ \\ -360\left(1-\cos\left|\sigma_f\right|\right)t^7+690\left(1-\cos\left|\sigma_f\right|\right)t^8+ \\ -600\left(1-\cos\left|\sigma_f\right|\right)t^9+200\left(1-\cos\left|\sigma_f\right|\right)t^{10} \end{pmatrix}^{1/2}$$

eq 27

The exponential coordinates σ_f represents the final orientation.
The left singular vectors (normal eigenvalues of $A^T A$) are given by :

$$U=\begin{pmatrix} \sigma_1 & U_{12} & U_{13} \\ \sigma_2 & U_{22} & U_{23} \\ \sigma_3 & U_{32} & U_{33} \end{pmatrix} \quad \textbf{eq 28}$$

The right singular vectors (normal eigenvalues of AA^T) can be written as :

$$V=\begin{pmatrix} \alpha\sigma_1 & V_{12} & 0 \\ \alpha\sigma_2 & V_{22} & V_{23} \\ \alpha\sigma_3 & V_{32} & V_{33} \end{pmatrix} \quad \textbf{eq 29}$$

Straight but tedious calculations lead to the proof of this Proposition.
The optimal curves in the ambient manifold assume analytical forms. Geodesics are straight lines, minimum acceleration curves are cubic polynomial curves and minimum jerk curves are fifth order polynomial curves, all parameterized by time. As the Singular Value Decomposition is a smooth operation, the projected curve on SO(3) is smooth.
When $tr(R)\neq 1$, the exponential coordinates are given by: $\sigma=\log R=\frac{\phi}{2\sin\phi}\left(R-R^T\right)$

Where ϕ satisfies

$$\cos\phi=\frac{1}{2}\left(tr(R)-1\right) \qquad \left|\phi\right|<\Pi \qquad \text{eq 30}$$

$$\hat{\omega}=R^T\dot{R} \quad \textbf{eq 31}$$

Straight derivations allow to obtain the linear and angular accelerations.
Thus, using the Euler-Poincaré equations, the desired inputs can be easily found.

$$f(u)=M\dot{v}+\omega xMv+b(.)$$
$$\tau(u)=J\dot{\omega}+\omega xJ\omega+v*Mv+\beta(.) \quad \textbf{eq 32}$$

A set of equations (depending on the propulsion) must be solved to find the reference control u.

<u>MOTION PLANNING BASED ON POLYNOMIAL INTERPOLATION</u>

Using the local parameterization of exponential coordinates, It can be seen that

$$\omega=\dot{\sigma}$$

Depending on the initial and final conditions, we can propose first, third or fifth degree polynomial variations .
<u>Proposition 7</u>:
- When the initial and final position d_i, d_f and orientation σ_i,σ_f are known : first order polynomial.

$$d(t)=\left(d_f-d_i\right)t+d_1$$
$$\sigma(t)=a_1t+a_0 \quad \textbf{eq 33}$$
$$a_0=\sigma_i \qquad a_1=\sigma_f-\sigma_i$$

- When the initial and final position d_i, d_f and orientation σ_i,σ_f with linear $\dot{d}_i,,\dot{d}_f$ and angular $\dot{\sigma}_i,\dot{\sigma}_f$ velocities are known : third order polynomial

$$d(t)=d_3t^3+d_2t^2+d_1t+d_0$$
$$\sigma(t)=a_3t^3+a_2t^2+a_1t+a_0 \quad \textbf{eq 34}$$

with

$$a_0=\sigma_i;\ a_1=\dot{\sigma}_i;$$
$$a_2=3\left(\sigma_f-\sigma_i\right)-\dot{\sigma}_f-2\dot{\sigma}_i;$$
$$a_3=-2\left(\sigma_f-\sigma_i\right)+\dot{\sigma}_f+\dot{\sigma}_i$$

- When the initial and final position d_i, d_f and orientation σ_i,σ_f with linear $\dot{d}_i,,\dot{d}_f,\ddot{d}_i,,\ddot{d}_f$, and angular $\dot{\sigma}_i,\dot{\sigma}_f,\ddot{\sigma}_i,\ddot{\sigma}_f$ velocities and accelerations are known : fifth order polynomial.

$$d(t)=d_5t^5+d_4t^4+d_3t^3+d_2t^2+d_1t+d_0$$
$$\sigma(t)=a_5t^5+a_4t^4+a_3t^3+a_2t^2+a_1t+a_0 \quad \textbf{eq 35}$$

with

$$a_0 = \sigma_i;\ a_1 = \dot{\sigma}_i;\ a_2 = 0.5\ddot{\sigma}_i$$

$$a_3 = 10(\sigma_f - \sigma_i) - \left(6\dot{\sigma}_i + 4\dot{\sigma}_f\right) + \frac{1}{2}\left(-3\ddot{\sigma}_i + \ddot{\sigma}_f\right)$$

$$a_4 = -15(\sigma_f - \sigma_i) + \left(8\dot{\sigma}_i + 7\dot{\sigma}_f\right) - \frac{1}{2}\left(-3\ddot{\sigma}_i + 2\ddot{\sigma}_f\right)$$

$$a_5 = 6(\sigma_f - \sigma_i) - 3\left(\dot{\sigma}_i + \dot{\sigma}_f\right) + \frac{1}{2}\left(-\ddot{\sigma}_i + \ddot{\sigma}_f\right)$$

eq 36

σ_i, σ_f represent respectively the initial and final orientation.
The parameters d_j are defined in the same way than the coefficients a_j (j=1,..,3).

MOTION GENERATION

If we find an input u(t) that achieves a desired motion in time 1, then $\dfrac{u\left(t/T_f\right)}{T_f^{2}}$ achieves the same motion in time T_f. This time/magnitude scaling property should be taken into account when applying the motion planning laws[4, 5, 6, 8]. The predicted arrival time can be calculated as the result of an optimization problem subject to dynamics and actuators constraints[4, 5].
In a general polynomial setting, the linear and angular velocities and acceleration can be written as :

$$v(t) = \sum_{i=1}^{m} \frac{d_i}{T_f^i} i t^{i-1} \qquad \omega(t) = \sum_{i=1}^{m} \frac{\sigma_i}{T_f^i} i t^{i-1}$$

$$a(t) = \sum_{i=2}^{m} \frac{d_i}{T_f^i} i(i-1) t^{i-2} \qquad \dot{\omega}(t) = \sum_{i=2}^{m} \frac{\sigma_i}{T_f^i} i(i-1) t^{i-2}$$

m=1, 3 or 5 depending on the order of the chosen polynomial.
Depending on the constraints, the predicted arrival time can be found analytically or as a solution of a nonlinear equation.

KINEMATICS CONSTRAINTS

The following motion generation problem can be formulated (with $t \in [0, T_f]$)

$$\begin{aligned} &\min \quad T_f \\ &s.t \quad |v| \le v_{\max} \quad |\omega| \le \omega_{\max} \\ &\qquad |a| \le a_{\max} \quad |\dot{\omega}| \le \dot{\omega}_{\max} \end{aligned}$$

eq 37

Depending on the degree of the chosen polynomial, we have different expressions:

First order

$$T_f = \max\left(\frac{A_d}{v_{\max}}, \frac{A_\sigma}{\omega_{\max}}\right)$$

eq 38

$$A_{d\,j} = \left|d_{f\,j} - d_{i_j}\right| \qquad A_\sigma = \left|\sigma_{f\,j} - \sigma_{i\,j}\right| \quad j = 1,..,3$$

Third order

$$T_f = \max\left(\frac{3A_d}{2v_{\max}}, \frac{3A_\sigma}{2\omega_{\max}}, \sqrt{\frac{6A_d}{a_{\max}}}, \sqrt{\frac{6A_\sigma}{\dot{\omega}_{\max}}}\right)$$

eq 39

Fifth order

$$T_f = \max\left(\frac{15A_d}{8v_{\max}}, \frac{15A_\sigma}{8\omega_{\max}}, \sqrt{\frac{10A_d}{\sqrt{3}a_{\max}}}, \sqrt{\frac{10A_\sigma}{\sqrt{3}\,\dot{\omega}_{\max}}}\right)$$

eq 40

DYNAMICS CONSTRAINTS

The following motion generation problem can be formulated :

$$\begin{aligned} &\min \quad T_f \\ &s.t \quad |v| \le v_{\max} \quad |\omega| \le \omega_{\max} \\ &\qquad |f(u)| \le f_{\max} \quad |\tau(u)| \le \tau_{\max} \end{aligned}$$

eq 41

where

$$f(u) = M\dot{v} + \omega x M v + b(.)$$

$$\tau(u) = J\dot{\omega} + \omega x J\omega + v * Mv + \beta(.)$$

Depending on the degree of the chosen polynomial, we have different expressions.

Let's suppose first that b(.)=β(.)=0.

First order polynomial

$$T_f = \max\left(\frac{\varphi_i}{f_{\max_i}}, \frac{\eta_i}{\tau_{\max_i}}, \frac{A_d}{v_{\max}}, \frac{A_\sigma}{\omega_{\max}}\right)_{i=1,3}$$

eq 42

where

$$\varphi_1 = A_{\sigma_2} m_3 - A_{\sigma_3} m_2$$
$$\varphi_2 = A_{\sigma_3} m_1 - A_{\sigma 1} m_3$$
$$\varphi_3 = A_{\sigma_1} m_2 - A_{\sigma_2} m_1$$
$$\eta_1 = A_{\sigma_2} J_3 - A_{\sigma_3} J_2 + A_{d_2} m_3 - A_{d_3} m_2$$
$$\eta_2 = A_{\sigma_3} J_1 - A_{\sigma 1} J_3 + A_{d_3} m_1 - A_{d1} m_3 \quad \textbf{eq 43}$$
$$\eta_3 = A_{\sigma_1} J_2 - A_{\sigma_2} J_1 + A_{d_1} m_2 - A_{d_2} m$$
$$m_i = \sum_{j=1}^{3} M_{ij} A_{d_j}$$
$$J_i = \sum_{j=1}^{3} J_{ij} A_{\sigma_j}$$

Third order polynomial

$$T_f = \max \begin{pmatrix} \sqrt{\dfrac{\varphi_i}{f_{\max_i}}}, \sqrt{\dfrac{\eta_i}{\tau_{\max_i}}}, \dfrac{3A_d}{2v_{\max}}, \\ \dfrac{3A_\sigma}{2\omega_{\max}}, \sqrt{\dfrac{6A_d}{a_{\max}}}, \sqrt{\dfrac{6A_\sigma}{\dot{\omega}_{\max}}} \end{pmatrix}_{i=1,3}$$

eq 44

where

$$\varphi_1 = 6(1-2\Xi_1)m_1 + 36\Xi_1^2(1-\Xi_1)^2\left(A_{\sigma_2} m_3 - A_{\sigma_3} m_2\right)$$
$$\varphi_2 = 6(1-2\Xi_2)m_2 + 36\Xi_2^2(1-\Xi_2)^2\left(A_{\sigma_3} m_1 - A_{\sigma 1} m_3\right)$$
$$\varphi_3 = 6(1-2\Xi_3)m_3 + 36\Xi_3^2(1-\Xi_3)^2\left(A_{\sigma_1} m_2 - A_{\sigma_2} m_1\right)$$
$$\eta_1 = 6(1-2\Psi_1)m_1 + 36\Psi_1^2(1-\Psi_1)^2\left(A_{\sigma_2} J_3 - A_{\sigma_3} J_2 + A_{d_2} m_3 - A_{d_3} m_2\right)$$
$$\eta_2 = 6(1-2\Psi_2)m_2 + 36\Psi_2^2(1-\Psi_2)^2\left(A_{\sigma_3} J_1 - A_{\sigma 1} J_3 + A_{d_3} m_1 - A_{d1} m_3\right)$$
$$\eta_3 = 6(1-2\Psi_3)m_3 + 36\Psi_3^2(1-\Psi_3)^2\left(A_{\sigma_1} J_2 - A_{\sigma_2} J_1 + A_{d_1} m_2 - A_{d_2} m_1\right)$$

with the coefficients Ξ_i and Ψ_i easily calculated by the software MAPLE. They depend on the dynamic model parameters and the difference between the initial and final position and orientation.

Fifth order polynomial

A 7th order polynomial equation must be solved numerically. The polynomial coefficients are calculated using Maple.

Let's suppose now that b(.)≠0 and β(.)≠0.

First order polynomial

A second order polynomial equation must be solved for each of the three forces and each of the three torques limitations.

For example, considering the limitation on the first force gives the following equation:

$$a_2 T_f^2 + a_1 T_f + a_0 = 0 \quad \textbf{eq 45}$$

$$a_2 = (\sigma_2 m_3 - \sigma_3 m_2)$$
$$a_1 = D_{V1} A_{d_1},$$
$$a_0 = \frac{\sigma_2(mg-B)}{\sigma_f}\sin\left(Arctg\left(\frac{\sigma_2\sigma_f}{\sigma_1\sigma_3}\right)\right) - \quad \textbf{eq 46}$$
$$\frac{\sigma_1\sigma_3(mg-B)}{\sigma_f^{\,2}}\left(1-\cos\left(Arctg\left(\frac{\sigma_2\sigma_f}{\sigma_1\sigma_3}\right)\right)\right) - F_{\max 1}$$

Five other equations have to be solved in the same way. Then the greatest value of all the 6 proposed times will be taken as the predicted arrival time T_f .

Third order polynomial

For the third and the fifth order polynomials, the problem is solved numerically. We are looking forward for a simple solution to be implemented.

SIMULATION RESULTS

The platform used for simulations is the AS200 airship (by Airspeed airships). It is a remotely piloted airship designed for remote sensing. It is a non rigid 6mlong, 1.4m diameter and $8.6m^3$ volume airship. In this paper, it is supposed to be fully actuated.

SINGULAR VALUE DECOMPOSITION

The initial and final positions are respectively $d(0) = \begin{pmatrix}0 & 0 & 0\end{pmatrix}^T$; $d(1) = \begin{pmatrix}8 & 10 & 12\end{pmatrix}^T$, for a normalized time $0 \le t \le 1$. End positions on SO(3) are given in exponential coordinates. The initial condition is $\sigma(0) = \begin{pmatrix}0 & 0 & 0\end{pmatrix}^T$ which corresponds to the body frame $\{M\}$ being parallel with the initial frame $\{F\}$ at t=0.

CHOICE OF THE METRIC

First, let's show the importance of the choice of the metric. The final condition $\sigma(1) = \begin{pmatrix}\frac{\pi}{6} & \frac{\pi}{3} & \frac{\pi}{2}\end{pmatrix}^T$ corresponds to a rotation of $\pi\frac{\sqrt{14}}{6}$ about the unit vector $\begin{pmatrix}\frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}}\end{pmatrix}^T$. For each figure, there are four subplots: the first one shows the three components of the exponential coordinates σ. The second one the components of the position vector, the third one the path in space. And finally the fourth one presents the linear velocity:

$$V_0 = \sqrt{u^2 + v^2 + w^2} \qquad \textbf{eq 47}$$

For a minimal jerk trajectory, with zero initial and final velocities and accelerations, four different simulations are presented :

A : First metric $W_1 = \alpha$ I (eq13)

$\alpha = 3$

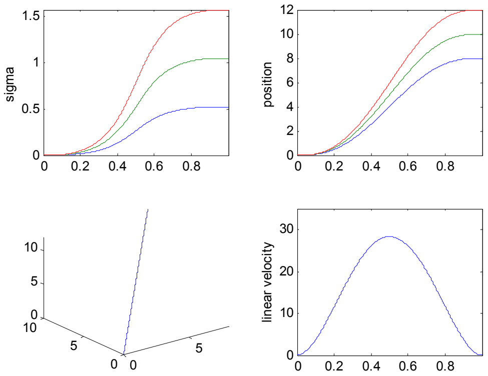

Figure 1

$\alpha = 10$

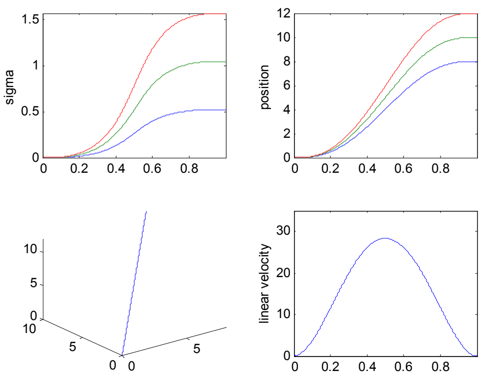

Figure 2

B : Second metric W_2 (inertia matrix) eq 8

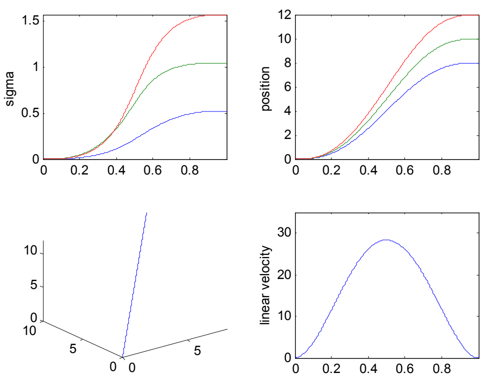

Figure 3

C : Third metric W_2 (eq 7)

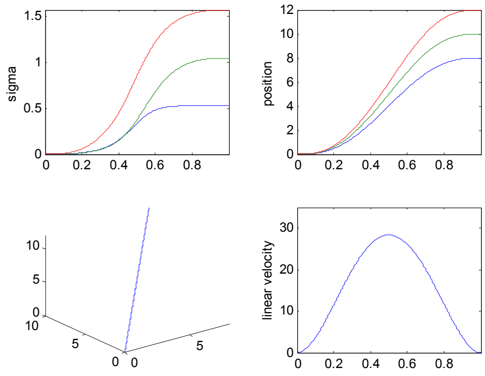

Figure 4

The chosen metric has obviously an effect on the variation of the exponential coordinates σ.

MINIMUM ENERGY, ACCELERATION AND JERK CURVES.

The first metric is chosen for these simulations. The predicted arrival time is $T_f = 10$. The final condition $\sigma(1) = \begin{pmatrix} \frac{\pi}{10} & \frac{\pi}{5} & \frac{\pi}{4} \end{pmatrix}^T$ corresponds to a rotation of $3\pi\frac{\sqrt{5}}{20}$ about the unit vector $\begin{pmatrix} \frac{2}{3\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{pmatrix}^T$. For each type of analytical trajectories, three figures are related:

Geodesic: Figure 5, Figure 6, Figure 7.

Minimal acceleration: Figure 8, Figure 9, Figure 10.

Minimal jerk: Figure 11, Figure 12, Figure 13.

For each figure (5, 8, 11), there are four subplots: the first one shows the three components of the exponential coordinates σ. The second one the components of the position vector, the third one the path in space. And finally the fourth one presents the linear velocity.

For each figure (6, 9, 12), there are four subplots: angular velocity and acceleration then linear velocity and acceleration.

For each figure (7, 10, 13), there are six subplots: the three first for the 3 components of the forces and the last three for the 3 components of the torques.

Geodesic

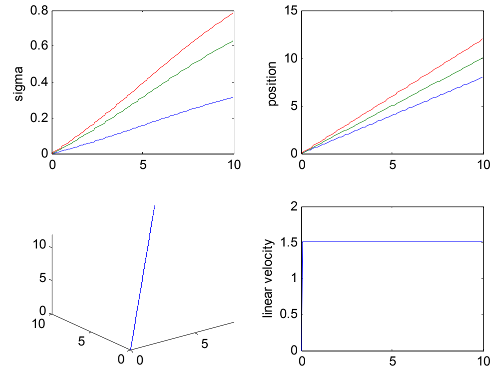

Figure 5

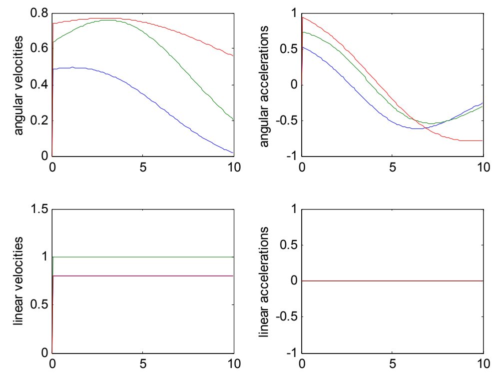

Figure 6

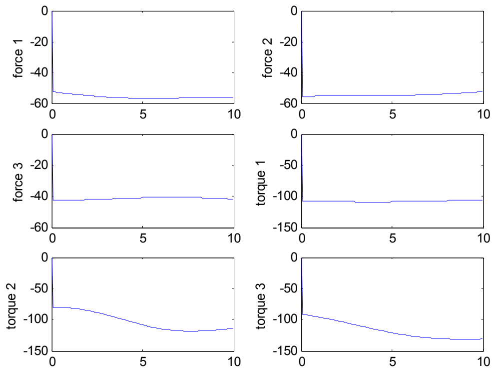

Figure 7

minimal acceleration

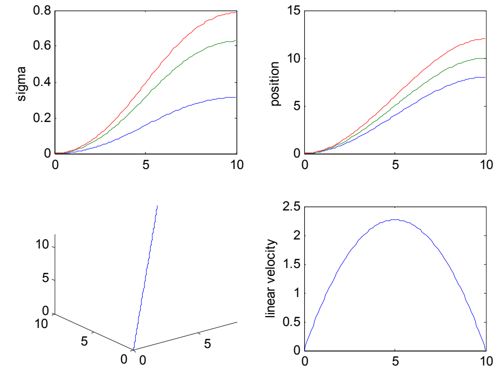

Figure 8

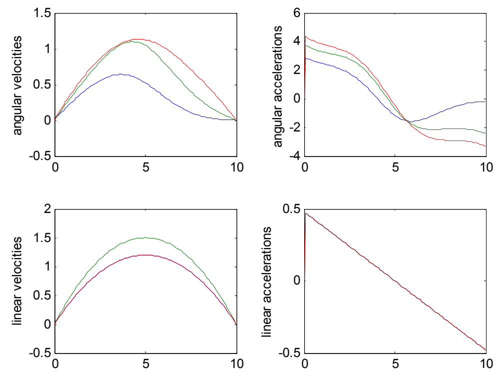

Figure 9

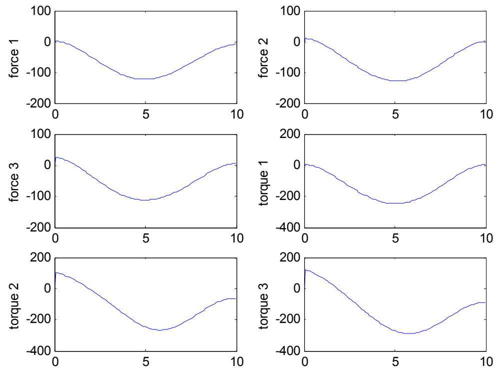

Figure 10

minimal jerk

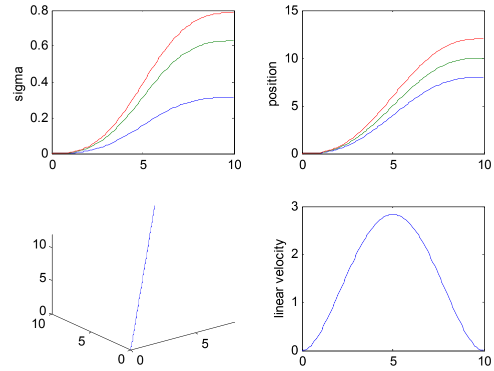

Figure 11

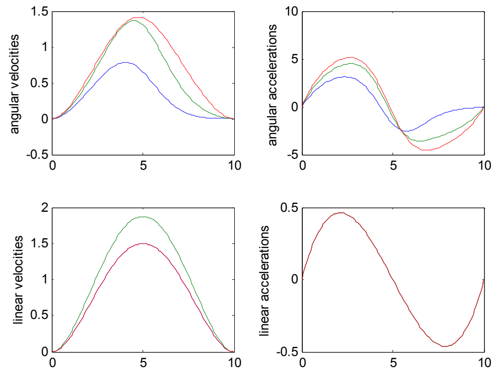

Figure 12

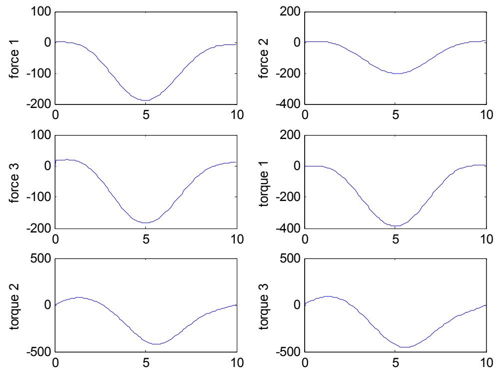

Figure 13

POLYNOMIAL INTERPOLATION

In this paragraph, due to the lack of space, we present only simulation results for the third order polynomial interpolation considering that b(.)≠0 and β(.)≠0. The respective limitations on the linear and angular velocities and accelerations are :

$v_{\max} = 13.4 m/s \qquad \omega_{\max} = 1.5 rad/s$

$\dot{v}_{\max} = 2.6 m/s^2 \qquad \dot{\omega}_{\max} = 5 rad/s^2$

While the limitations on the forces and torques are :

$F_{\max} = (100 \quad 100 \quad 100) N$

$\Gamma_{\max} = (100 \quad 200 \quad 200) Nm$

The predicted arrival time is $T_f = 15.6 s$

Figures 14, 15 and 16 present the simulation results for the same initial and final positions and orientations than above.

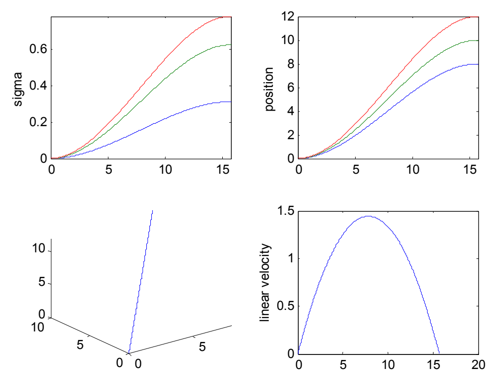

Figure 14

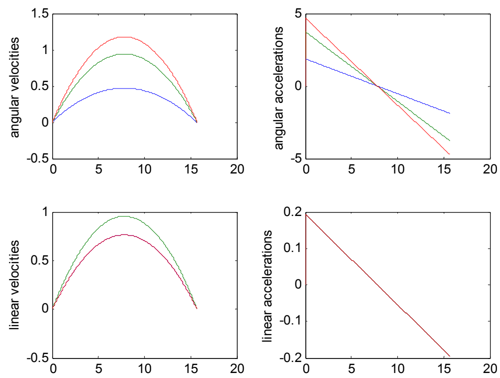

Figure 15

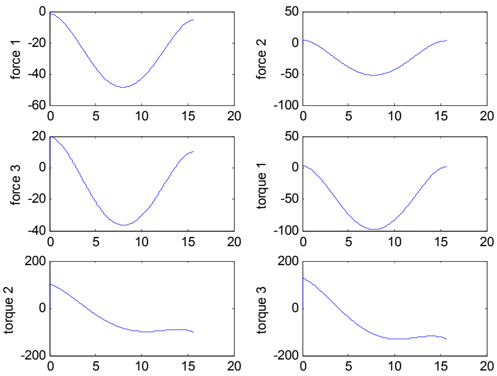

Figure 16

CONCLUSIONS

Once the path has been calculated in the Earth fixed frame, motion must be investigated and reference trajectories determined taking into account actuators constraints. Trajectory design can be formulated as a curve describing the time history of the vehicle displacement. The role of the trajectory generator is to generate a feasible time trajectory for the UAV. Feasible means that the trajectory fulfills the dynamics and actuators constraints. The challenge is to determine the best predicted arrival time in light of the capabilities of the vehicle.
A generalization of this work is trajectory generation for underactuated systems.

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