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A SENSITIVITY ANALYSIS OF THE COMPUTED TORQUE TECHNIQUE

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Abstract: Feedback design research has classically ties with Lyapunov stability theory and with the classical theory of dynamical systems. This work presents a sensitivity analysis of the computed torque technique applied to robotic control, using the bifurcation approach.

I- INTRODUCTION

Theory of non linear dynamical systems has attracted considerable interest in recent years. Some issues such as stability, non linear oscillations, bifurcations and presence or absence of chaos are relevant. One can modify system properties or behavior using the adequate controller. Work on bifurcation control is a very interesting development. In this case, one has available the concept of imperfection sensitivity which is useful for the study of the corruption of systems near the onset of a dynamical instability [3].

In this paper, the problem of sensitivity to gain variations is considered. We employ the bifurcation approach to give a specified neighborhood avoiding the chaotic motion. A particular control can significantly reduce the amplitude of a bifurcated solution, or significantly enhance its stability over a nontrivial parameter range. We will show that the perturbed problem admits a solution if the perturbations are continuous and bounded in some neighbourhoods.

II - COMPUTED TORQUE TECHNIQUE

The dynamics of a multilink articulated robot manipulator can be characterised by a set of nonlinear and coupled second-order differential equation : $\Gamma = D(q)\, q'' + h(q,q')$ (1)

where Γ are n x 1 external applied torques for joint actuators, n is the number of degrees of freedom, q, q' and q" are respectively n x 1 joint positions, velocities and accelerations. h(q,q') is the gravitational, Coriolis and Centrifugal force vector and D(q) is the positive definite n x n inertia matrix. To control the manipulator, the following control law is proposed:

$$\Gamma = \Gamma^d + \Delta\Gamma \quad (2)$$

where $\Gamma^d = \mathcal{D}(q^d)\, q''^d + \mathcal{h}(q^d, q'^d)$ (3)

and $\Delta\Gamma = \mathcal{D}(q^d)\,[K_v(t)\,(q'^d - q') + K_p(t)\,(q^d - q) +$

$$K_I(t) \int_0^t (q^d(\tau) - q(\tau))\, d\tau] \quad (4)$$

where $\mathcal{D}$ and $\mathcal{h}$ are estimates of D and h respectively, K_v, K_p and K_I are n x n diagonal gain matrices with K_{vj}, K_{pj} and K_{Ij} on the diagonals. The desired trajectory (q^d, q'^d) of the manipulator is given by the trajectory planning system [4].

III - SENSITIVITY ANALYSIS

Since one does not have access to the exact inverse dynamics, the linearization and the decoupling will not be exact. This is manifested by uncertain feedback terms that may be handled using multivariable robust control techniques. The i^{th} joint has as closed loop dynamics:

$$x'''(q,t) + K_v(t)\, x''(q,t) + K_p(t)\, x'(q,t) + K_I(t)\, x(q,t) = R(t,q,q',q'') \quad (5)$$

with

$$x(q,t) = \int_0^t (q^d(\tau) - q(\tau))\, d\tau \qquad \text{and } x(0) = x'(0) = 0$$

$$R(t,q,q',q'') = \mathcal{D}^{-1}(q^d)\,[\,(\mathcal{D}(q^d) - D(q))\, q'' + \mathcal{h}(q^d,q'^d) - h(q,q')]$$

The functions $\mathcal{D}$, D, $\mathcal{h}$, h and R are assumed to be C^1 uniformly bounded functions when q,q' and q" are bounded. The problem to tackle with may be stated as:

"May a small variation of the gain matrices lead to a similar behavior of the system or to a completely different one ?"

Let $\varepsilon_{kv}(t)$, $\varepsilon_{kp}(t)$ and $\varepsilon_{ki}(t)$ n-vectors representing the perturbations of the gain matrices. We would like to study the existence, unicity and stability of the solution q(t) associated with the pertubed problem:

$$x'''(q,t) + (K_v(t) + I\varepsilon_{kv}(t))x''(q,t) + (K_p(t) + I\varepsilon_{kp}(t))x'(q,t) \quad (6)$$
$$+ (K_i(t) + I\varepsilon_{ki}(t))x(q,t) = R(t,q,q',q''),\ x(0) = x'(0) = 0,\ t \in J$$

First, let's have an abstract formulation of the problem.

a) Abstract formulation of the problem

Let's define the mathematical spaces where the problem solutions lie. Let $Z = C(J, \mathbb{R}^n)$ the set of continuous functions on J. Let us define a norm on Z such that

$\forall z \in Z,\ \|z\|_\infty = \text{Sup}\, \|z(t)\| \quad t \in J$

Thus $(Z, \|.\|_\infty)$ is a Banach space.

Let $SO = C^3(J, \mathbb{R}^n)$ the set of three times continuously differentiable functions on J and let us define a norm on SO such that

$\forall y \in SO,\ \|y\|_{SO} = \|y\|_\infty + \|y'\|_\infty + \|y''\|_\infty + \|y'''\|_\infty$

Thus $(SO, \|.\|_{SO})$ is a Banach space.

Let us define the linear operator on S by:

$$LX = \frac{d^3X}{dt^3} + K_v(t)\frac{d^2X}{dt^2} + K_p(t)\frac{dX}{dt} + K_I(t)X$$

where the set S is defined as:

$S = \{X \in SO, X(0) = X'(0) = 0, \|X\|_\infty \le r\}$

and Λ is the family of all continuous pertubations on J,

$\varepsilon \in \Lambda;\ \varepsilon(t) = (\varepsilon_{kv}(t), \varepsilon_{kp}(t), \varepsilon_{ki}(t))$

We will study the existence of nontrivial solutions x of:

$$x'''(q,t) + K_v(t)\, x''(q,t) + K_p(t)\, x'(q,t) + K_i(t)\, x(q,t) = \quad (7)$$
$$= -\,[I\varepsilon_{ki}(t)x(q,t) + I\varepsilon_{kp}(t)x'(q,t) + I\varepsilon_{kv}(t)x''(q,t)] + \mu\, R(t,q,q',q'')$$
$$x(0) = x'(0) = 0 \qquad t \in J \quad \mu \in \mathbb{R},$$

Let 's define $\quad h: J \times \mathbb{R}^n \times \mathbb{R} \times \Lambda \to \mathbb{R}$

$$h(t, x, \mu, \varepsilon) = -\,[I\varepsilon_{kv}(t))x''(q,t) + I\varepsilon_{kp}(t))x'(q,t + I\varepsilon_{ki}(t))x(q,t)\,] + \mu\, R(t,q,q',q'') \quad (8)$$

and the Nemytskii operator $\mathcal{h}$ associated to h defined as:

$\mathcal{h}:\ S \times \mathbb{R} \times \Lambda \to Z$

$\qquad X, \mu, \varepsilon \to \mathcal{h}(X,R,\varepsilon)(t) = h(t, X(t), \mu, \varepsilon)$

$\varepsilon \in \Lambda,\ \mu \in \mathbb{R},\ X \in \text{dom}L,\ t \in J$

$\mathcal{h}$ satisfies the following properties:

1) $\mathcal{h}$ is continuous with respect to X, μ and ε.

2) $\mathcal{h}(0,0,0) = 0$

Problem (6) is equivalent ot the abstract formulation:

$LX = \mathcal{h}(X,\mu,\varepsilon);\ \varepsilon \in \Lambda,\ \mu \in \mathbb{R},\ X \in \text{dom}L,\ t \in J$ (9)

A solution of eq (9) is also solution of eq (6) and reciprocally. The operator L has the following properties:

1) Ker L is spanned by the three linearly independent solutions of the homogeneous problem associated to the non perturbed problem:

$X2' = -A_1^T X2 \qquad X2(0) = (0,0,x''(0)) \qquad (10)$

where T denotes the transpose;

$$A_1(t) = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -K_I(t) & -K_p(t) & -K_v(t) \end{pmatrix} \text{ and } X2 = \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix}$$

Ker L= span $\{X_{01}, X_{02}, X_{03}\}$ and dim Ker L = 3.

2) $\ell \in$ Im L if and only if :

$$\int_0^T (\ell(t), X_{01})\,dt = \int_0^T (\ell(t), X_{02})\,dt = \int_0^T (\ell(t), X_{03})\,dt = 0$$

so Codim Im L = 3. This relation is given by the Fredholm alternative.

3) L is an indice 0 Fredholm operator. As Ker L and CoKer L have finite dimension, they admit supplementaries, more, they are closed. So we may define projections on each of these sets: $c_1, c_2, c_3 \in \mathbb{R}$, $\quad P_0: S \rightarrow S$

$$P_0X(t) = c_1 X_{01}(t) \int_0^T (X(\tau), X_{01}(\tau))\,d\tau +$$

$$c_2X_{02}(t) \int_0^T (X(\tau), X_{02}(\tau))\,d\tau + c_3X_{03}(t) \int_0^T (X(\tau), X_{03}(\tau))\,d\tau$$

$I - P_1: Z \rightarrow Z$

$$P_1X(t) = c_1 X_{01}(t) \int_0^T (X(\tau), X_{01}(\tau))\,d\tau +$$

$$+c_2X_{02}(t) \int_0^T (X(\tau), X_{02}(\tau))\,d\tau + c_3 X_{03}(t) \int_0^T (X(\tau), X_{03}(\tau))\,d\tau$$

where the constants c_1, c_2, c_3 are such that Ker L basis is normalised. P_0 is a projection of S onto Ker L such that : IM P_0 = Ker L.

P_1 is a projection of Z onto Im L such that: Ker $(I-P_1)$ = IM L.

The Fredholm alternative shows that the equation: $LX = \ell$, $\ell \in Z$ admits a solution if and only if: $(I - P_1)\ell = 0$.

In this case, a unique solution $X = X(\ell) \in S$ exists and is such that: $\quad P_0X = 0$

The complementary projections of P_0 and $(I-P_1)$ are defined in the following sets:

$I-P_0: S \rightarrow \mathcal{S} \qquad S = \text{Ker}\, L \oplus \mathcal{S}$

$P_1 : Z \rightarrow \mathcal{Z} \qquad Z = \text{Im}\, L \oplus \mathcal{Z}$

$\forall\, X \in S,\ \exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R},\ v \in \mathcal{S}$ such that:

$X = \alpha_1X_{01} + \alpha_2X_{02} + \alpha_3X_{03} + v$

$\forall\, \ell \in Z,\ \exists \ell_1 \in \text{Im}\, L,\ \ell_2 \in \mathcal{Z}$ such that: $\ell = \ell_1 + \ell$

Let us define the continuous integral operator K that kernel is the Green function associated to problem (6) :

K: $\ell \in P_1Z = \text{Im}\, L \rightarrow \quad K(\ell) \in (I-P_0)\,S$.

Using this operator, we may propose the following sensitivity analysis.

b) Sensitivity analysis

First we may propose the following theorem:

Theorem 1: $X = KP_1\ell$ is the unique solution of :

$LX = P_1\ell \qquad$ and $\qquad P_0 K P_1 \ell = 0$

and another solution of problem (14) is given by:

$X = \alpha_1X_{01} + \alpha_2X_{02} + \alpha_3X_{03} + K P_1 \ell;\ \alpha_i \in \mathbb{R}. (11)$

Proof of Theorem1 mav be found in [1].

For $X \in$ Dom L, using relation (11), the abstract problem (9) is equivalent to :

$v = K P_1 (h(\alpha_1X_{01} + \alpha_2X_{02} + \alpha_3X_{03} + v, \mu, \varepsilon)) \qquad (12)$

$0 = (I-P_1) (h(\alpha_1X_{01} + \alpha_2X_{02} + \alpha_3X_{03} + v, \mu, \varepsilon)) \qquad (13)$

Relation (12) is called auxiliary equation and (13) bifurcation equation. These relations are dependent one from each other. They form a system of equations to be solved simultaneously. Using this formulation, we may propose the following theorem:

Theorem 2: Assuming that the gain matrices are such that: $\quad K_{pj}^2 = \dfrac{K_{Ij}}{K_{vj}}$ there exist neighborhoods V1 of v=0 in $(I-P_0)$ S, V2 of α=0 in $\mathbb{R}^3$, V3 of ε=0 in Λ and finally V4 of μ=0 in $\mathbb{R}$ such that if $(\alpha, \varepsilon, \mu) \in$ V2 x V3 x V4, the auxiliary equation (12) has a unique solution $v^*(\alpha,\varepsilon,\mu) \in$ V1 if and only if $(\alpha, \varepsilon, \mu)$ is a solution of the bifurcation equation (13): v^* : V2 x V3 x V4 $\rightarrow$ V1 with $v^*(0,0,0) = 0$.

Proof of Theorem 2 may be found in [1].

When the gains are constant, the problem solution are the α_i function of μ (i=1,...,4).

Theorem 3: Under the assumptions of Theorem 2:

1 - If $\mu = 0$, there exist neighborhoods V2 of α=0 in $\mathbb{R}^3$, V3 of ε=0 in Λ with V3= V3' $\cup$V3", V3' and V3" are two connexe disjoint subsets, (0$\in$ V3') such that :

i) if $\varepsilon \in$ V3', then problem (6) has a unique solution X*

ii) if $\varepsilon \in$ V3", then problem (6) has a infinite number of solutions bifurcating from the unique solution X*.

2 - If $\mu \neq 0$, there exist neighborhoods W2 of α=0 in $\mathbb{R}^3$, V3 of ε=0 in Λ with W3= W3' $\cup$W3", W3' and W3" are two connexe disjoint subsets, (0$\in$ W3') such that

i) if $\varepsilon \in$ W3', then problem (6) has at least one solution in V1.

ii) if $\varepsilon \in$ W3", then problem (6) has no solution in V1.

Proof of Theorem 3 may be found in [1].

VI - CONCLUSIONS

Manipulator's control system based on computer torque technique incorporates a model of the manipulator dynamics. The nominal torque computed using this mathematical model, does not reflect the effects of unknown loadings and uncertainty in modelling the parameters. The problem of stabilization using bifurcation approach via direct state feedback is considered and more precisely sensitivity is investigated through the study of the auxiliary and bifurcation equations. The proposed theorems consider the development of sufficient conditions for the stabilization of a manipulator, in relation with sensitivity.

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