ANALYSIS OF THE ROBOTIC COMPUTED TORQUE TECHNIQUE

Yasmina Bestaoui
Laboratoire d'Automatique de Nantes, UA CNRS 823
Ecole Centrale de Nantes/Université de Nantes, 1 rue de la Noë, 44072 Nantes, France

Djamila Benmerzouk
Département de Mathématiques, Institut des Sciences Exactes
Université de Tlemcen, BP 119, 13000 Tlemcen, Algeria

Abstract. : This work presents a stability and sensitivity analysis of the computed torque technique applied to robotic control, using the regular perturbation theory. Perturbation theory is a general tool for multiple-time, scale system and robustness studies. The proposed theorem considers the development of sufficient conditions for the robust stabilization of a manipulator. In relation with stability, a sensitivity analysis to gain variations is proposed.

I- INTRODUCTION

In this paper, we study the stabilizability problem for robot manipulator control. This system is a smooth nonlinear one that is affine in the control:

$$x' = F(x) + \Sigma_{i=1}^{p} \Gamma_i G_i(x))$$

$$\text{and } F(0) = 0;\ x \in \mathbb{R}^n,\ \Gamma \in \mathbb{R}^m \qquad (1)$$

x is the space vector, Γ the control vector.

First, constant gains were used to stabilize this system [9]. Then, piecewise linear feedback and other discontinuous types of feedback have been proposed in [10] to regulate non linear systems. It was proved that if the system is analytic and completely controllable, then there exists a piecewise analytic stabilizing feedback. [1] showed that there exists an ordinary stabilizing feedback that is continuous for every $x \neq 0$ in a neighbourhood of $0 \in \mathbb{R}^n$. [3] proposed a stability analysis using continuous PD and PID gains varying linearly. [10] studied constant and variable PD non-linear regulator gains. We finally mention [8] who employed Lie algebra and [12] who used the Lyapunov method.

Although, this work is based on [10], the proposed stability analysis is quite different. Samson et al studied a PD regulator, we use a PID one. Furthermore, the stability theorem and the proof are different and the sensitivity analysis is original.

The remainder of this paper is organized as follows. The mathematical model is given in the following section and the computed torque technique in Section III. Sections IV and V present respectively the stability and sensitivity analysis, consisting in two original theorems, Finally, Appendices A and B present the proofs of the two mentioned theorems.

Notations: $\|.\|$ represents the vectorial euclidean norm, $\|.\|_0$ is the associated matricial norm, (.,.) is the scalar product in $\mathbb{R}^n$ and I is the identity matrix.

II - MATHEMATICAL MODEL

The dynamics of a multilink articulated robot manipulator, excluding the actuator dynamics, gear friction and backlash, can be characterised by a set of nonlinear and coupled second-order differential equation :

$$\Gamma = D(q)\, q'' + h(q,q') \qquad (2)$$

where n is the number of degrees of freedom, Γ are n x 1 external applied torques for joint actuators, q, q' and q" are respectively n x 1 joint angles, velocities and accelerations, h(q,q') is the gravitational, Coriolis and Centrifugal force vector and D(q) is an n x n inertia matrix.

III - COMPUTED TORQUE TECHNIQUE

Industry generally uses classical techniques like PD or PID controllers with constant parameters to operate the robots that it employs. However, the litterature [3,5,6] has been investigating how to increase the speed and tracking accuracy of the manipulators.The computed torque technique sets the basis upon which much of the present literature on robotic trajectory control is based. In this method we compute the necessary torques based on the inertial dynamics of the manipulator. In the control problem, preceeding the robot dynamics, an inverse model is incorporated. This inverse model calculates the torques needed for the reference trajectory. A great advantage is that the whole system (inverse model+robot) has an almost linear behavior.

To control the manipulator, the following control law is proposed:

$$\Gamma = \Gamma^d + \Delta\Gamma \qquad (3)$$

where

$$\Gamma^d = D(q^d(t))\, q''^d + h(q^d(t), q'^d(t)) \qquad (4)$$

and

$$\Delta\Gamma = \hat{D}(q^d(t))\,[K_v(t)\,(q'^d(t) - q'(t)) + K_p(t)\,(q^d(t) - q(t)) + K_I(t)\int_0^t (q^d(\tau) - q(\tau))\,d\tau] \quad (5)$$

where $\hat{D}$ and $\hat{h}$ are estimates of D and h respectively, $K_v(t)$, $K_p(t)$ and $K_I(t)$ are n x n diagonal gain matrices with $K_{vj}(t)$, $K_{pj}(t)$ and $K_{Ij}(t)$ on the diagonals (j=1,..,n), t denotes the time, $t \in [0,T]$, T being the predicted arrival time. The desired trajectory (q^d, q'^d) of the manipulator is given by the trajectory planning system [5,6]. The PID controller is applied in order to obtain some sensitivity improvements.

IV - EXISTENCE AND STABILITY ANALYSIS

4 - 1 Problem formulation

Since one does not have access to the exact inverse dynamics, the linearization and the decoupling will not be exact. This is manifested by uncertain feedback terms that may be handled using multivariable robust control techniques.

The i^{th} joint has as closed loop dynamics:

$$x'''(q,t) + K_v(t)\,x''(q,t) + K_p(t)\,x'(q,t) + K_I(t)\,x(q,t) = R(t,q,q',q'') \quad (6)$$

with

$$x(q,t) = \int_0^t (q^d(\tau) - q(\tau))\,d\tau$$

$$x'(q,t) = e(q,t) = q^d(t) - q(t)$$
$$x''(q,t) = e'(q,t) = q'^d(t) - q'(t)$$
$$x'''(q,t) = e''(q,t) = q''^d(t) - q''(t) \quad (7)$$
and $x(q,0) = x'(q,0) = 0$

$$R(t,q,q',q'') = \hat{D}^{-1}(q^d(t))[(\hat{D}(q^d(t)) - D(q(t)))q''(t) + \hat{h}(q^d(t),q'^d(t)) - h(q(t),q'(t))] \quad (8)$$

where $x(q,t) = (x_1(q,t),\ldots, x_n(q,t))^T$ n dimensional vector

The functions $\hat{D}(q^d)$, $D(q)$, $\hat{h}(q^d,q'^d)$ and $h(q,q')$ are assumed to be C^1 so they are uniformly bounded when q,q' and q'' are bounded, R(t,q,q',q'') is also assumed to be a C^1 function uniformly bounded with respect to its arguments; $t \geq 0$. The nonlinear vectorial function R cannot be treated as an external disturbance. It represents a disturbance of the globally linearized error dynamics which is caused by modeling uncertainties, parameter variations and external disturbances. The multivariable approach then revolve around the design of a controller such that the complete closed-loop system is stable in some suitable sense, e.g uniformly ultimately bounded, globally asymptotically stable, etc, for a given class of nonlinear functions.

Remark: If all the dynamics are exactly known for the control of a manipulator, the computed torque controller is known to be asymptotically stable in following a desired trajectory [5].

4 - 2 Existence and stability

The aim of this paragraph is to present a stability analysis of the computed torque technique, giving some sufficient conditions on the gain parameters. Robustness is studied when disturbances are acting on the system; it is based on adequate choices of the feedback gains.

First, a definition taken from [10] is presented.

Definition of the r-admissibility: Let e(q,t) be a vectorial application of class C^k, $k \geq 1$ from an open subset Ω of $\mathbf{R}^n \times \mathbf{R}$ to $\mathbf{R}^n$. e(q,t) is an r-admissible function on the set $C_{r,T}$ during the time interval [0,T] if and only if the function F(q,t) =(e(q,t),t) is a C^k class diffeomorphism (i.e its reciprocal is also a C^k class-bijection,) from $C_{r,T}$ onto the closed sphere B(0,r) x [0,T] centered at 0, of radius r.

Theorem 1: Existence, unicity and stability

If the gain matrices fulfill the following conditions:

H1) $K_p(t)$, $K_v(t)$ and $K_I(t)$ locally integrable

*) Each coefficient of $K_p(t)$, $K_v(t)$ and $K_I(t)$ is Borel measurable on $C_{r,T}$

**) Sup ($\|K_p(t)\|_0, \|K_v(t)\|_0, \|K_I(t)\|_0$) is Lebesgue integrable on each compact interval $I_\tau = [0,\tau]$ i.e

$$\int_0^t \text{Sup}\,(\|K_p(t)\|_0, \|K_v(t)\|_0, \|K_I(t)\|_0)\,dt < \infty$$
$$t \in [0,T] \quad (9)$$

H2) $K_p(t)$ is positive definite and uniformly bounded on I_τ, and its coefficients are derivable,

$$\text{Sup}_{t \in I_\tau}(\|K_p(t)\|_0) < \infty \quad I_\tau = [0,\tau]$$
and $\text{sup}_{t \in I_\tau}(\|K_v(t)\|_0) < \infty \quad (10)$

H3) There exists a constant γ such that:

$$K_{vj}(t)\,K_{pj}(t) - K_{Ij}(t) + K'_{pj}(t) = \gamma,\ j=1,\ldots,n \quad (11)$$

H4) $$\|x''(q,0)\| + \frac{\sup(\|R(t,q,q',q'')\|)}{\sup(\|K_v(t)\|_0)} \leq$$
$$\leq \min(1, \inf(\|K_p(t)\|_0)).\, r \quad (12)$$

We have then the following results:

i) Existence and unicity of the solution of problem (6) : The solution X(q,t) exists and is unique on $C_{r,T}$.

ii) Stability of the solution:

$$\forall\, t \in I_\tau,\ \|X(q,t)\| < r \quad \text{where } X = (x, x', x'')^T \quad (13)$$

Moreover, the solution X(q,t) is exponentially stable in the neighbourhood of 0.

The proof of this theorem is adapted from perturbations techniques [1,2,7]. It may be found in Appendix A.

Remark: The constant γ is introduced to avoid cancellation of the integral gain K_{Ij} when the derivative gain K_{vj} and K'_{pj} are null.

Theorem 1 allows us to analyze the effects of every component appearing in the differential equation on the stability and the asymptotic behaviour of the error e(q,t). In fact, it shows that if the initial error $\|e(q,0)\|$ is small enough, then there exists a minimal nonlinear gain such that if $\|K_p(t)\|$ is greater than this gain, then $\|e(q,t)\|$ stays inside the sphere $B(0,r)$. A constant gain, when large enough, may be sufficient to guarantee stability of equation (1) in a domain that size increases with the size of K_p. Large constant gains do also present drawbacks, like sensitivity to noise or high energy in the control torques.

V - SENSITIVITY ANALYSIS

5 - 1 Problem formulation

The problem to tackle with may be stated as:
"May a small variation of the gain matrices lead to a similar behavior of the system or to a completely different one ?"

Let $\varepsilon_{kv}(t)$, $\varepsilon_{kp}(t)$ and $\varepsilon_{ki}(t)$ n-vectors representing the perturbations of the elements of the diagonal gain matrices. We would like to study the existence, unicity and stability of the solution q(t) associated with the pertubed problem:

$$x'''(q,t) + (K_v(t)+I\varepsilon_{kv}(t))x''(q,t) + (K_p(t)+I\varepsilon_{kp}(t))x'(q,t) + (K_i(t)+I\varepsilon_{ki}(t))\, x(q,t) = R(t,q,q',q'') \qquad (14)$$

$$x(q,0) = x'(q,0) = 0 \qquad , t \in J,\ J= [0,T] \qquad (15)$$

5 - 2 Existence, unicity and stability

a) Abstract formulation of the problem

Let $Z=C(J,\mathbb{R}^n)$ the set of continuous functions on J. Let us define a norm on Z such that

$$\forall\ z \in Z,\ \|z\|_\infty = \mathrm{Sup}\ \|z(t)\| \quad t \in J \qquad (16)$$

Thus $(Z, \|.\|_\infty)$ is a Banach space[7]

Let $SO = C^3(J,\mathbb{R}^n)$ the set of three times continuously differentiable functions on J and let us define a norm on SO such that

$$\forall\ y \in SO,\ \|y\|_{SO}= \|y\|_\infty + \|y'\|_\infty + \|y''\|_\infty + \|y'''\|_\infty \qquad (17)$$

Thus $(SO, \|.\|_{SO})$ is a Banach space.

Let us define the linear operator on S by:

$$Lx = \frac{d^3x}{dt^3} + K_v(t)\,\frac{d^2x}{dt^2} + K_p(t)\,\frac{dx}{dt} + K_I(t)x \qquad (18)$$

where the set S is defined as:

$$S = \{x \in SO,\ x(q,0)= x'(q,0)= 0,\ \|x\|_\infty \le r\} \qquad (19)$$

and Λ is the family of all continuous pertubations on J, $\varepsilon \in \Lambda$

$$\varepsilon(t) = (\varepsilon_{kv}(t), \varepsilon_{kp}(t), \varepsilon_{ki}(t)) \qquad (20)$$

Let $h: J \times (\mathbb{R}^n)^2 \times \Lambda \rightarrow \mathbb{R}^n$

$$h(t,x,R,\varepsilon)= -[I\varepsilon_{kv}(t)x''(q,t)+I\varepsilon_{kp}(t)x'(q,t) + I\varepsilon_{ki}(t)x(q,t)] + R(t,q,q',q'') \qquad (21)$$

and the Nemytskii operator $\mathfrak{h}$ associated to h defined as:

$\mathfrak{h}$: $S \times Z \times \Lambda \rightarrow Z$

$$x \rightarrow (\mathfrak{h}(x,R,\varepsilon)\,(t) = h(t,x(q,t),R(t,q,q',q''), \varepsilon) \qquad (22)$$

$\varepsilon \in \Lambda,\ R \in Z,\ x \in S,\ t \in J$

$\mathfrak{h}$ satisfies the following properties:

1) $\mathfrak{h}$ is continuous with respect to x, R and ε.

2) $\mathfrak{h}(0,0,0) = 0 \qquad (23)$

Problem (14) is equivalent to the abstract formulation:

$$Lx = \mathfrak{h}(x,R,\varepsilon) \quad \varepsilon \in \Lambda,\ R \in Z,\ x \in \mathrm{dom}L\ ,\ t \in J \qquad (24)$$

A solution of problem (14) is also solution of problem (24) and reciprocally.

b) Theorem 2:

If the gains matrices fulfill conditions H1, H2, H3 and H4 of Theorem 1 and

H5) $\varepsilon_{kv}(t)$, $\varepsilon_{kp}(t)$, $\varepsilon_{ki}(t)$ are bounded on each open subinterval of J.

Then the solution x0(q,t) of problem (14) exists, is unique and stable in a neighbourhood of 0 in $C_{r,T}$.

The proof of Theorem2 may be found in Appendix B.

Remark: Extension to $[0, +\infty[$. The results obtained in Theorem2 may be applied to $x0(\varepsilon)$ and on each subinterval of $[0, +\infty[$. with the conditions

a) $x0(\varepsilon)$ is infinitely r-admissible on $C_r = \mathrm{Sup}\ C_{r,T} \quad t \in [0, +\infty[$.

b) $\|[\frac{dx0}{dq}(q,t)]^{-1}\|_0$ is uniformly bounded on C_r.

c) $\|\frac{dx0}{dt}(q,t)\|_0$ is uniformly bounded on C_r.

d) $\varepsilon_{kv}(t)$, $\varepsilon_{kp}(t)$, $\varepsilon_{ki}(t)$ are uniformly continuous and bounded on $\mathbb{R}^n$.

VI - CONCLUSIONS

Manipulator's control system based on computed torque technique incorporates a model of the manipulator dynamics. The nominal torque, computed using this mathematical model, does not reflect the effects of unknown loadings and uncertainty in modelling the parameters. An approach is presented in this paper, considering this problem. We propose a stability analysis of the computed torque technique using a PID regulator, with nonlinear varying gains.

Use of nonlinear gains provides an additional degree of freedom in the design of control scheme, as an alternative to the use of a constant gain. It may allow the overall performance of the system to be improved. The reason is that a constant gain may be turned with respect to the most unfavourable case, while a non linear gain can be designed so as to vary with the robot configuration, and assume large values only when it is really needed. It is also interesting to investigate if a small change of the regulator gains has a small effect on the behavior of the robot.

REFERENCES

[1] Artstein Z. "Stabilization with relaxed controls" Non linear analysis, vol 7, 1983, pp1163-1173

where

$M = \| x''(0)\| + \sup_{(q,t)\in Cr,t} \|R(t,q,q',q'')\|/m$ (A15)

thus ε_1 is uniformly bounded onto $C_{r,\tau}$ by M, for $\tau \le T$.

3- ε_1 satisfy equation (A3) and the matrices $K_p(t)$, $K_v(t)$ and $K_I(t)$ verify assumption (H3). We have then:

$$x'' = \varepsilon_1 - K_p(t)\, x \qquad \text{(A16)}$$

using matricial notation, we may write:

$$\frac{dX_1}{dt} = -A2(t)\, X_1 + \begin{pmatrix} 0 \\ \varepsilon_1 \end{pmatrix} \qquad \text{(A17)}$$

where:

$$A2(t) = \begin{pmatrix} 0 & I \\ K_p(t) & 0 \end{pmatrix} \qquad \text{(A18)}$$

with $X_1(q,0) = 0$ and $X_1 = (x,x')^T$

The homogeneous problem associated with problem (A17) is given by:

$$\frac{dX_1}{dt} = -A2(t)\, X_1 \quad \text{with } X(q,0) = 0 \qquad \text{(A19)}$$

It is equivalent to the equation:

$$X_1^T X_1' = -X_1^T\, A2(t)\, X_1 \qquad \text{(A20)}$$

Let

$$a = \min(1, \inf_{t\in I\tau} \|K_p(t)\|) \qquad \text{(A21)}$$

we have:

$$X_1^T\, A2(t)\, X1 \ge a\, \|X_1(t)\|^2 \qquad \text{(A22)}$$

The solution of problem (A17) satisfy the inequality

$$\|X_1(q,t)\| \le \|x''(q,0)\| \exp(-at) \qquad \text{(A23)}$$

so, we can deduce that the non-homogeneous problem solution satisfy:

$$\|X_1(q,t)\| \le \frac{1}{a} \sup_{t\in I\tau} \|\varepsilon_1(t)\| \exp(-at) \qquad \text{(A24)}$$

so, $\|X_1(q,t)\| \le r \exp(-at)$ (A25)

Thus,

$\|X_1(q,t)\| \le r$, for each $t \in I_\tau$. (A26)

4) We demonstrated that $X_1(q,t)$ is uniformly bounded by r on each interval I_τ. More, $\varepsilon_1(t)$ is also uniformly bounded on I_τ. Thus, the solution $X_1(q,t)$ exists and is unique on the interval [0,T[, by continuity, x(q,t) exists and is unique on I.

Moreover, the solution of problem (6) verify (A25). Thus, the solution x(q,t) of problem (6) is exponentially stable in the neighbourhood of 0.

APPENDIX B: Proof of Theorem 2

First let us recall the following theorem [7].

Leray-Schauder Theorem: Let S be a banach space and $U(.,\varepsilon)$ an operator defined from S to S depending on a parameter ε, $\varepsilon \in [\varepsilon_1,\varepsilon_2] \supset \{0\}$ (for some chosen ε_1 and ε_2). Assume that:

1) $U(y,\varepsilon) = 0$ for an $\varepsilon \in [\varepsilon_1,\varepsilon_2]$, $\forall\, y \in S$ (B1)

2) $U(y,\varepsilon)$ is continuous and compact for $y \in S$, $\varepsilon \in [\varepsilon_1,\varepsilon_2]$ (B2)

3) $U(y,\varepsilon)$ is uniformly bounded on ε, on each bounded subset of S.

4) There exists an a-priori estimation, independent of ε, for the fixed points of $U(y,\varepsilon)$.

Then, there exists a continuum of solution of equation $y = U(y,\varepsilon)$ when ε takes all values in the interval $[\varepsilon_1,\varepsilon_2]$.

The proof of Theorem2 is broken into two steps:

A- First step: Existence and unicity of the solution on I

*) If $\varepsilon_{kv}(t)=0$, $\varepsilon_{kp}(t)=0$ and $\varepsilon_{ki}(t)=0$, then there is no perturbation acting on the dynamical system and in this case, problem(14) is equivalent to problem(6) and Theorem1 applies. The solution x(q,t) is unique and stable on $C_{r,T}$.

*) For an $\varepsilon\in\Lambda$, let us define:

$$x0(q,t) = x(q,t) + y(t) \qquad \text{(B3)}$$

Thus, problem(24) is equivalent to:

$$Ly = \hbar(x+y,0,\varepsilon) = \hbar 1(y,\varepsilon) \qquad \text{(B4)}$$

where $y \in$ dom1L with

$$\text{dom1L} = \{y\in S,\ y(0)=y'(0)=y''(0)=0;\ \|y\| \le r\} \qquad \text{(B5)}$$

and $\varepsilon\in\Lambda$.

As the homogeneous problem Lx=0 admits as unique solution the trivial one, using the Fredholm alternative we may conclude that L is invertible. Let us call L^{-1} the inverse operator of L, the closed graph theorem [7] shows that L^{-1} is bounded. $\alpha 0$ is the superior bound of L^{-1}. To find a solution of problem (B4) is to find a fixed point of the operator

$$U(.,\varepsilon) = L^{-1}(\hbar 1(.,\varepsilon)),\ \varepsilon\in\Lambda. \qquad \text{(B6)}$$

Let us use the Leray-Schauder theorem and first verify if its assumptions are fulfilled:

1) When the dynamical system is not perturbed (i.e $\varepsilon=0$) then $U(y,0) = 0$.

2) continuity of $U(y,\varepsilon)$ on y.

First, let us define $(y_n)_{n\in N}$ a serie such that $(y_n)\to y$

Let us show that: $U(y_n,\varepsilon) \to U(y,\varepsilon)$ $\varepsilon \in \Lambda$.
we have:

$\| h1(t,y_n(t),\varepsilon) - h1(t,y(t),\varepsilon) \| =$

$\| \varepsilon_{kv}(t)\,(y''_n(t)-y''(t)) + \varepsilon_{kp}(t)\,(y'_n(t)-y'(t)) + \varepsilon_{kI}(t)\,(y_n(t)-y(t)) \| \leq$

$\|\varepsilon_{kv}(t)\|\,\|(y''_n(t)-y''(t))\| + \|\varepsilon_{kp}(t)\|\,\|(y'_n(t)-y'(t))\| + \|\varepsilon_{kI}(t)\|\,\|(y_n(t)-y(t))\|$ (B7)

So, we may write:

$\operatorname{Sup} \| h1(t,y_n(t),\varepsilon) - h1(t,y(t),\varepsilon) \| \leq \operatorname{Sup}(\|\varepsilon_{kv}\|_\infty, \|\varepsilon_{kp}\|_\infty, \|\varepsilon_{kI}\|_\infty) \| y_n(t)-y(t) \|$ $t \in J$. (B8)

and thus:

$\| h1(t,y_n(t),\varepsilon) - h1(t,y(t),\varepsilon) \|_\infty \leq \varepsilon_2 \| y_n - y \|_S$ (B9)

where:

$\varepsilon_2 \geq \operatorname{Sup}(\|\varepsilon_{kv}\|_\infty, \|\varepsilon_{kp}\|_\infty, \|\varepsilon_{kI}\|_\infty)$ (B10)

This implies that:

$\| h1(y_n,\varepsilon) - h1(y,\varepsilon) \|_Z \leq \varepsilon_2 \| y_n - y \|_S$ (B11)

We can deduce then that:

$\| L^{-1} (h1(y_n,\varepsilon) - h1(y,\varepsilon)) \|_S \leq \alpha 0\, \varepsilon_2 \| y_n - y \|_S$ $\varepsilon \in \Lambda$. (B12)

Then, $U(y,\varepsilon)$ is uniformly bounded.

Let us suppose additionally that $\alpha 0\, \varepsilon_2 < 1$. In this case, the operator $U(y,\varepsilon)$ is a contraction on S.

The operator L^{-1} is continuous and compact, $h1(y,\varepsilon)$ is also continuous so $L^{-1} (h1(y_n,\varepsilon))$ is compact and continous, i.e completely continuous.

3) Boundedness of U on ε

Let us show that:

$\forall \zeta > 0,\ \exists \eta(\zeta) > 0$ /

$\| \varepsilon - \varepsilon' \|_\Lambda < \eta(\zeta) \Longrightarrow \| U(y,\varepsilon) - U(y,\varepsilon') \|_S < \varepsilon$ (B13)

we have:

$\| h1(y,\varepsilon) - h1(y,\varepsilon') \|_Z \leq \| \varepsilon - \varepsilon' \|_\Lambda \| y \|_S$ $\forall \varepsilon,\varepsilon' \in \Lambda$. (B14)

that implies:

$\| U(y,\varepsilon) - U(y,\varepsilon') \|_S \leq \alpha 0 \| \varepsilon - \varepsilon' \|_\Lambda \| y \|_S < \zeta$ $\forall \varepsilon,\varepsilon' \in \Lambda$. (B15)

If we choose:

$0 < \eta(\zeta) \leq \zeta / \alpha 0$ (B16)

Thus, the operator U is uniformly bounded with respect to ε.

4) $y - U(y,\varepsilon) = 0 \Longrightarrow 0 \geq \| y \|_S - \| U(y,\varepsilon) \|_S$ (B17)

As we have:

$\| U(y,\varepsilon) \|_S \leq \alpha 0\, \varepsilon_2 (\| y \|_S + \| X \|_S)$ (B18)

we may write:

$0 \geq (1 - \alpha 0\, \varepsilon_2) \| y \|_S - \| X \|_S$ (B19)

Thus:

$$\| y \|_S \leq \frac{\| X \|_S}{(1 - \alpha 0\, \varepsilon_2)} = r_0 \qquad \text{(B20)}$$

r_0 is independent from the perturbations.

All the assumptions of the Leray-Schauder Theorem are fulfilled by the operator $U(y,\varepsilon)$, $\forall y \in S$, $\forall \varepsilon \in [0,\varepsilon_2]$. Thus, its conclusions hold. The solution of the problem $U(y,\varepsilon) = y$, $y \in S$ and $\varepsilon \in \Lambda$, admits a continuum of solutions $y(\varepsilon) \in S$ unique where S takes values in $[0, \varepsilon_2]$. $y(\varepsilon)$ is continuous and $\lim_{\varepsilon \to 0} y(\varepsilon) = 0$

$y(\varepsilon)$ is r-admissible on S as $x(t)$ is r-admissible on S and y verifies equation (24), $\| y \| \leq r$, $\forall t \in J$, and thus the solution of problem (14) exists and is unique.

B- Second step: stability of the solution.

As the initial condition of problem(14) verifies hypothesis H4 of Theorem1 and y(t) verifies the relation , the solution $y(\varepsilon)$ verifies:

$\forall \varepsilon \in \Lambda$, $\forall t \in J$, $\| y(\varepsilon) \| \leq r$,

Thus $y(\varepsilon)$ is stable on $C_{r,T}$, on a neighbourhood of 0.

The solution $y(\varepsilon) \in B(0,r)$, the results hold for $q(\varepsilon)$ solution of problem (14), with $\varepsilon \in \Lambda$, $\varepsilon \in [0, \varepsilon_2]$ exists, is unique on $C_{r,T}$, and stable in a neighbourhood of 0.

[2] **Benmerzouk D.** "Problème d'évitement d'obstacles et étude de la commande dynamique d'un robot manipulateur à axes rigides" Ms Thesis, Math. Dept, University of Tlemcen, Algeria, May 1991
[3] **Bestaoui Y.** "Decentralised PD and PID robotic regulators " IEE Proceedings on Control Theory and Applications, vol 136, pt D, n°4, July 1989, pp133-145
[4] **Brauer F.** "Nonlinear differential equations with forcing terms" Proceedings of the American Mathematical Society, vol n°5, 1964, pp758-765
[5] **Craig J.J** "Introduction to Robotics, mechanics & control" Addison Wesley Publ, 1986
[6] **Dombre E., Khalil W.** "Modélisation et commande des robots" Hermes Publ, Paris, 1988
[7] **Hale J.** "Ordinary differential equations" Wiley Interscience,NY, 1969
[8] **Nijmeier H., Van Der Schaft A.J** "Non linear dynamical control systems" Springer Verlag, New York, 1990
[9] **Paul R.P** "Robot manipulators, mathematics, programming and control" MIT Press, 1981
[10] **Samson C., Le Borgne M., Espiau B.** "Robot control, the task function approach" Oxford Science Publications, 1991
[11] **Sontag E.D** "Non linear regulation: the piecewise linear approach" IEEE Transactions on Automatic Control, vol 26, 1981, pp346-358.
[12] **Wijesoma S.W, Richards R.J** "Robust following of robots using computed torque structure with VSS" International Journal of Control, vol 52, pp935-962, 1990

APPENDIX A: PROOF OF THEOREM 1

With matrix notation, we obtain for the non homogeneous problem:

$$X' = A(t)\,X + B(t) \quad X(0) = (\,0\;\;0\;\;x''(0))^T \qquad \text{(A1)}$$

where the matrix A is given by

$$A(t) = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -K_I(t) & -K_p(t) & -K_v(t) \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 \\ 0 \\ R(t,q,q',q'') \end{pmatrix} \qquad \text{(A2)}$$

1- The problem is described by equation (A1). $R(t,q,q',q'')$ is continuously differentiable and the matrices $K_p(t)$, $K_v(t)$ and $K_I(t)$ verify assumption (H1). The initial conditions being$(0,0,x''(0))$ with hypothesis (H4), the solution of (A1) exists and is locally unique on some interval $I_\tau=[0,\tau[;\ \tau < T$.

To show that the solution of (A1) exists globally on I_τ, we'll prove that x is uniformly bounded on some interval I_τ by r and x' and x" are uniformly bounded by constants independent of r.

2- Let's introduce the auxiliary vectoriel function ε_1 defined as:

$$\varepsilon_1 : C_{r,T} \longrightarrow \mathbf{R}^n$$

$$\varepsilon'_1 = -K_v(t)\,\varepsilon_1 + R(t,q,q',q'')$$

$$\text{with} \quad \varepsilon_1(0) = x''(q,0) \qquad \text{(A3)}$$

We must verify that ε_1 is uniformly bounded onto $C_{r,T}$. The system (A3) is equivalent to:

$$\varepsilon'_1 = -K_v(t_1)\,\varepsilon_1 - (K_v(t) - K_v(t_1))\,\varepsilon_1 + R(t,q,q',q'')$$

$$\text{with} \quad \varepsilon_1(0) = x''(q,0) \qquad \text{(A4)}$$

where $t_1 \in I_\tau$.

The solution of the following problem:

$$\varepsilon'_1 = -K_v(t_1)\,\varepsilon_1 \ \text{ with } \varepsilon_1(0) = x''(q,0) \qquad \text{(A5)}$$

verify

$$\varepsilon_1(t)\,\varepsilon^{-1}{}_1(s) = \exp(-K_v(t_1)\,(t-s))\,x''(q,0) - \int_s^t (\exp(-K_v(t_1)(t-\sigma))(K_v(\sigma)-K_v(t_1))\varepsilon_1(\sigma)\varepsilon^{-1}{}_1(s)d\sigma) + \int_s^t (\exp(-K_v(t_1)(t-\sigma))\,R(\sigma,q,q',q'')\,d\sigma)$$

$$0 \le s \le t \le \tau \qquad \text{(A6)}$$

Let

$$m = \operatorname{Inf}_{t \in I\tau} \|K_v(t)\|_0 \qquad \text{(A7)}$$

From [7], we have:

$$\|\exp(-K_v(t_1)(t-s))\,x''(q,0)\| \le \|x''(q,0)\|\exp(-m\,(t-s)) \quad \text{(A8)}$$

$K_v(t)$ being locally integrable on I_τ and $R(t,q,q',q'')$ being uniformly bounded on $C_{r,T}$, if

$$\|t - \sigma\| < \eta(\varepsilon), \text{ for each } \varepsilon>0 \qquad \text{(A9)}$$

then ε_1 is such that:

$$\|\varepsilon_1(t)\,\varepsilon^{-1}{}_1(s)\| \le \|x''(q,0)\|\exp(-m\,(t-s)) + \varepsilon \int_s^t (\exp(-m(t-\sigma))\,\|\varepsilon_1(\sigma)\varepsilon^{-1}{}_1(s)\|\,d\sigma) + \sup_{t\in It}\|R(t,q,q',q'')\| \int_s^t \|\exp(-K_v(t_1)(t-s))\|\,d\sigma) \quad \text{(A10)}$$

Let

$$Y_1(t) = \exp(mt)\,\varepsilon_1(t) \qquad \text{(A11)}$$

using the preceeding inequality (A10), we have:

$$\|Y_1(t)\,Y_1{}^{-1}(s)\| \le \|x''(q,0)\| + \frac{1}{m}\sup_{(q,t)\in Cr,t}\|R(t,q,q',q'')\| + \varepsilon \int_s^t \|Y_1(\sigma)\|\|Y^{-1}{}_1(s)\|\,d\sigma \qquad \text{(A12)}$$

Cronwall lemma [7] implies that:

$$\|Y_1(t)\,Y^{-1}{}_1(s)\| \le [\,\|x''(q,0)\| + \frac{1}{m}\sup_{(q,t)\in Cr,t}\|R(tq,q',q'')\|]\exp(-\varepsilon(t-s))$$

$$0 \le s \le t \le \tau \le T. \qquad \text{(A13)}$$

and thus

$$\|\varepsilon_1(t)\,\varepsilon^{-1}{}_1(s)\| \le M \qquad \text{(A14)}$$